

Ozawa's class \mathcal{S} for locally compact groups and unique prime factorization of group von Neumann algebras

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June 7, 2019

Abstract

We study class \mathcal{S} for locally compact groups. We characterize locally compact groups in this class as groups having an amenable action on a boundary that is small at infinity, generalizing a theorem of Ozawa. Using this characterization, we provide new examples of groups in class \mathcal{S} and prove a unique prime factorization theorem for group von Neumann algebras of products of locally compact groups in this class. We also prove that class \mathcal{S} is a measure equivalence invariant.

1 Introduction

Class \mathcal{S} for countable groups was introduced by Ozawa in [Oza06]. A countable group Γ is said to be in *class \mathcal{S}* if it is exact and it admits a map $\eta : \Gamma \rightarrow \text{Prob}(\Gamma)$ satisfying

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

for all $g, h \in \Gamma$. Equivalently, class \mathcal{S} can be characterized as the class of all groups that admit an amenable action on a boundary that is *small at infinity* (see [Oza06, Theorem 4.1]). Groups in class \mathcal{S} are also called *bi-exact*.

Class \mathcal{S} is used in, among others, [Oza04; Oza06; OP04; CS13; PV14; CI18; CdSS16] to prove rigidity results for group von Neumann algebras of countable groups. In [Oza04], Ozawa proved that the group von Neumann algebra $L(\Gamma)$ is solid when Γ belongs to class \mathcal{S} . This implies in particular that for Γ icc, nonamenable and in class \mathcal{S} , the group von Neumann algebra $L(\Gamma)$ is *prime*, i.e. $L(\Gamma)$ does not decompose as a tensor product $M_1 \otimes M_2$ for non-type I factors M_1 and M_2 .

In [OP04], Ozawa and Popa proved the first unique prime factorization results for von Neumann algebras using groups in this class. Among other results, they showed that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of nonamenable, icc groups in class \mathcal{S} , then $L(\Gamma) \cong L(\Gamma_1) \otimes \cdots \otimes L(\Gamma_n)$ remembers the number of factors n and each factor $L(\Gamma_i)$ up to amplification, i.e. if $L(\Gamma) \cong N_1 \otimes \cdots \otimes N_m$ for some prime factors N_1, \dots, N_m , then $n = m$ and (after relabeling) $L(\Gamma_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$. Subclasses of class \mathcal{S} were used in [CS13; PV14; HV13] to prove rigidity results on crossed product von Neumann algebras $L^\infty(X) \rtimes \Gamma$.

Examples of countable groups in class \mathcal{S} are amenable groups, hyperbolic groups (see [Ada94]), lattices in connected simple Lie groups of real rank one (see [Ska88, Proof of Théorème 4.4]), wreath products $B \wr \Gamma$ with B amenable and Γ in class \mathcal{S} (see [Oza06]) and $\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z})$ (see [Oza09]). Moreover, class \mathcal{S} is closed under measure equivalence (see [Sak09]). Examples of groups not belonging to class \mathcal{S} are product groups $\Gamma \times \Lambda$ with Γ nonamenable and Λ infinite, nonamenable inner amenable groups and nonamenable groups with infinite center.

In this paper, we study class \mathcal{S} for locally compact groups. We provide a characterization of groups in this class similar to [Oza06, Theorem 4.1], we provide new examples of groups in this class and we prove a unique prime factorization result for group von Neumann algebras of locally compact groups. We also prove that class \mathcal{S} is a measure equivalence invariant.

Let G be a locally compact second countable (lcsc) group. We denote by $\text{Prob}(G)$ the space of all Borel probability measures, i.e. the state space of $C_0(G)$. The precise definition of class \mathcal{S} for locally compact groups is now as follows.

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Definition A. Let G be a lcsc group. We say that G is in *class \mathcal{S}* (or *bi-exact*) if G is exact and if there exists a $\|\cdot\|$ -continuous map $\eta : G \rightarrow \text{Prob}(G)$ satisfying

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0 \quad (1.1)$$

uniformly on compact sets for $g, h \in G$.

In [BDV18] this property without the exactness condition was called *property (S)*. Note that the definition was slightly different: the image of the map η above was in the space $\mathcal{S}(G) = \{f \in L^1(G)^+ \mid \|f\|_1 = 1\}$ instead of $\text{Prob}(G)$. However, we prove in Proposition 3.1 that this is equivalent. It is also worthwhile to note that it is currently unknown whether there are groups with property (S) that are not exact.

Examples of lcsc groups in class \mathcal{S} include amenable groups, groups acting continuously and properly on a tree or hyperbolic graph of uniformly bounded degree and connected, simple Lie groups of real rank one with finite center. Proofs of these results can be found in [BDV18, Section 7]. It is easy to prove that groups not in class \mathcal{S} include product groups $G \times H$ with G nonamenable and H non-compact, nonamenable groups G with non-compact center and nonamenable groups G that are *inner amenable at infinity*, i.e. for which there exists a conjugation invariant mean m on G such that $m(E) = 0$ for every compact set $E \subseteq G$ (see also Proposition 4.8).

We prove a version of [Oza06, Theorem 4.1] for locally compact groups in class \mathcal{S} , i.e. we characterize groups in class \mathcal{S} as groups acting amenably on a boundary that is *small at infinity*. Given a locally compact group G , we denote by $C_b^u(G)$ the algebra of *bounded uniformly continuous* functions on G , i.e. the bounded functions $f : G \rightarrow \mathbb{C}$ such that

$$\|\lambda_g f - f\|_\infty \rightarrow 0 \quad \text{and} \quad \|\rho_g f - f\|_\infty \rightarrow 0$$

whenever $g \rightarrow e$. Here, λ and ρ denote the left and right regular representations respectively, i.e. $(\lambda_g f)(h) = f(g^{-1}h)$ and $(\rho_g f)(h) = f(hg)$. We define the compactification $h^u G$ of the group G as the spectrum of the following algebra

$$C(h^u G) \cong \{f \in C_b^u(G) \mid \rho_g f - f \in C_0(G) \text{ for all } g \in G\}$$

and denote by $\nu^u G = h^u G \setminus G$ its boundary. The compactification $h^u G$ is equivariant in the sense that both actions $G \curvearrowright G$ by left and right translation extend to continuous actions $G \curvearrowright h^u G$. It is also *small at infinity* in the sense that the extension of the action by right translation is trivial on the boundary $\nu^u G$. It is moreover the universal equivariant compactification that is small at infinity, in the sense that for every equivariant compactification \overline{G} that is small at infinity, the inclusion $G \hookrightarrow \overline{G}$ extends to a continuous G -equivariant map $h^u G \rightarrow \overline{G}$.

The characterization of groups in class \mathcal{S} now goes as follows.

Theorem B. *Let G be a lcsc group. Then, the following are equivalent*

- (i) G is in class \mathcal{S} ,
- (ii) the action $G \curvearrowright \nu^u G$ induced by left translation is topologically amenable,
- (iii) the action $G \curvearrowright h^u G$ induced by left translation is topologically amenable,
- (iv) the action of $G \times G$ on the spectrum of $C_b^u(G)/C_0(G)$ induced by left and right translation is topologically amenable.

The two novelties in the proof of this result are the proof of (iii) and the method we used to prove the implication (iv) \Rightarrow (i). Indeed, in the original proof of Ozawa in the countable setting, it was proven that G belongs to class \mathcal{S} if and only if there exists a u.c.p map $\theta : C_r^*(G) \otimes_{\min} C_r^*(G) \rightarrow B(L^2(G))$ satisfying $\varphi(x \otimes y) - \lambda(x)\rho(y) \in K(L^2(G))$, where λ and ρ denote the representations of $C_r^*(G)$ induced by the left and right regular representation respectively. This is however no longer true for locally compact groups. Indeed, for all connected groups G , the reduced C^* -algebra $C_r^*(G)$ is nuclear and hence a map θ as above always exists.

Denote by $\beta^{lu} G$ the left-equivariant Stone-Ćech compactification of G , i.e. the spectrum of the algebra $C_b^{lu}(G)$ of bounded left-uniformly continuous functions on G . The action $G \curvearrowright G$ by left-translation extends uniquely to a continuous action $G \curvearrowright \beta^{lu} G$. Moreover, $\beta^{lu} G$ is the universal left-equivariant compactification of G in the sense that every left- G -equivariant continuous map $G \rightarrow X$ to any compact space X with continuous action $G \curvearrowright X$ extends uniquely to a G -equivariant continuous map $\beta^{lu} G \rightarrow X$. We also prove the following characterization of groups in class \mathcal{S} .

Theorem C. *Let G be a lcsc group. Then G belongs to class \mathcal{S} if and only if G is exact and there exists a Borel map $\eta : G \rightarrow \text{Prob}(\beta^{lu}G)$ satisfying*

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

In the proof of this theorem, we will see that it is precisely the exactness of G that allows us to construct the required map $\tilde{\eta} : G \rightarrow \text{Prob}(G)$ from a map $\eta : G \rightarrow \text{Prob}(\beta^{lu}G)$. This was implicitly observed before in [BO08, Chapter 15] for countable groups.

Using Theorem B, we prove the following new examples of locally compact groups in class \mathcal{S} . In [Cor17], Cornuier introduced a notion of wreath products for locally compact groups. See (4.7) on page 15 for a short recapitulation of this notion and the notation used in this article. The following result is a locally compact version of [Oza06, Corollary 4.5].

Theorem D. *Let B and H be lcsc groups, X a countable set with a continuous action $H \curvearrowright X$ and $A \subseteq B$ be a compact open subgroup. If B is amenable, all stabilizers $\text{Stab}_H(x)$ for $x \in X$ are amenable and H belongs to class \mathcal{S} , then also the wreath product $B \wr_X^A H$ belongs to class \mathcal{S} .*

A notion of measure equivalence for locally compact groups was introduced by S. Deprez and Li in [DL14]. By [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)] exactness is preserved under this notion of measure equivalence. More recently, this notion was studied in more detail in [KKR17; KKR18]. It was proved that two lcsc groups G and H are measure equivalent if and only if they admit essentially free, ergodic pmp actions on some standard probability space for which the cross section equivalence relations are stably isomorphic. Using this characterization, we were able to prove the following result. For countable groups it was proven by Sako in [Sak09].

Theorem E. *The class \mathcal{S} is closed under measure equivalence.*

As a consequence of this theorem, we have for instance that $\mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{Z})$ belong to class \mathcal{S} . Indeed, $\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z})$ is a lattice in both $\mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \text{SL}_2(\mathbb{Z})$. Hence, the latter two are measure equivalent with $\mathbb{Z}^2 \rtimes \text{SL}_2(\mathbb{Z})$, which belongs to class \mathcal{S} by [Oza09].

In [BDV18], the author proved together with Brothier and Vaes that the group von Neumann algebra $L(G)$ is solid whenever G is a locally compact group in class \mathcal{S} . In particular, when $L(G)$ is also a nonamenable factor, it follows that $L(G)$ is prime. Combining Theorem B with the unique prime factorization results of Houdayer and Isono in [HI17] along with the generalization [AHHM18, Application 4] by Ando, Haagerup, Houdayer, and Marrakchi, we were able to obtain the following unique prime factorization result for (tensor products of) such group von Neumann algebras.

Theorem F. *Let $G = G_1 \times \cdots \times G_m$ be a direct product of locally compact groups in class \mathcal{S} whose group von Neumann algebras $L(G_i)$ are nonamenable factors. If*

$$L(G) \cong N_1 \overline{\otimes} \cdots \overline{\otimes} N_n$$

for some non-type I factors N_i , then $n \leq m$. Moreover, all factors N_i are prime if and only if $n = m$ and in that case (after relabeling) $L(G_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$.

We prove this theorem by proving that for groups G in class \mathcal{S} , the group von Neumann algebra $L(G)$ belongs to the class $\mathcal{C}_{(AO)}$ introduced in [HI17].

It is worthwhile to note that for many locally compact groups G , the group von Neumann algebra $L(G)$ is amenable or even type I. For instance, the group von Neumann algebra of a connected lcsc group is always amenable by [Con76, Corollary 6.9]. However, the following group G due to Suzuki provides an example of a locally compact group whose group von Neumann algebra $L(G)$ is a nonamenable type II_∞ factor.

Example G (Suzuki). Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on \mathbb{F}_2 by flip of the generators. Then the compact group $K = \prod_{k \in \mathbb{N}} \mathbb{Z}_2$ acts on the infinite free product $H = \star_{k \in \mathbb{N}} \mathbb{F}_2$ by letting the k^{th} component of K flip the generators in the k^{th} component of H . The semi-direct product $G = H \rtimes K$ satisfies the conditions of [Suz16, Proposition] with $K_n = \prod_{k=n+1}^\infty \mathbb{Z}_2$ and $L_n = (\star_{k=0}^n \mathbb{F}_2) \rtimes K$. Hence, by [Suz16, section on Group von Neumann algebras], its group von Neumann algebra is a nonamenable factor of type II_∞ . Moreover, G belongs to class \mathcal{S} since G is measure equivalent to H and H belongs to class \mathcal{S} .

Furthermore, certain classes of groups acting on trees have nonamenable group von Neumann algebras by [HR19, Theorem C and D]. Also, [Rau19b, Theorem E and F] would provide conditions on such groups under which $L(G)$ is a nonamenable factor. In particular, for every $q \in \mathbb{Q}$ with $0 < q < 1$ [Rau19b, Theorem G] would provide examples of groups in class \mathcal{S} for which the group von Neumann algebra is a nonamenable factor of type III $_q$. However, due to a mistake in [Rau19b, Lemma 5.1] in that paper, there is a gap in the proofs of these results (see also [Rau19a, p 20]), and it is currently not completely clear whether these results hold as stated there.

Acknowledgments The author would like to thank Arnaud Brothier and Stefaan Vaes for interesting discussions and helpful comments.

2 Preliminaries and notation

Throughout this article, we will assume all groups to be locally compact and second countable. We denote by λ_G the left Haar measure on such a group G . All topological spaces are assumed to be locally compact and Hausdorff. All actions $G \curvearrowright X$ are assumed to be continuous.

Let X be a locally compact space. We denote by $M(X)$ the space of complex Radon measures on X . We can equip this space with the norm of total variation, or alternatively with the weak* topology when viewing it as the dual space of $C_0(X)$. The Borel structure from both topologies agree. We denote by $M(X)^+$ the space of positive Radon measures and $\text{Prob}(G)$ the space of Radon probability measures. Suppose that a group G acts on X , then for $g \in G$ and $\mu \in M(X)$ we denote by $g \cdot \mu$ the measure defined by $(g \cdot \mu)(E) = \mu(g^{-1}E)$ for all Borel sets $E \subseteq X$.

2.1 Topological amenability

We recall from [Ana02] the notion of topological amenability for actions of locally compact groups.

Definition 2.1. Let G be a lcsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. We say that $G \curvearrowright X$ is (*topologically*) *amenable* if there exists a net of weakly* continuous maps $\mu_i : X \rightarrow \text{Prob}(G)$ satisfying

$$\lim_i \|g \cdot \mu_i(x) - \mu_i(gx)\| = 0 \tag{2.1}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

By [Ana02, Proposition 2.2], we have the following equivalent characterization.

Proposition 2.2. *Let G be a lcsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, the following are equivalent*

- (i) $G \curvearrowright X$ is amenable
- (ii) There exists a net $(f_i)_i$ in $C_c(X \times G)^+$ satisfying

$$\lim_i \int_G f_i(x, s) \, ds = 1$$

uniformly on compact sets for $x \in X$ and

$$\lim_i \int_G |f_i(x, g^{-1}s) - f_i(gx, s)| \, ds = 0 \tag{2.2}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

Remark 2.3. Obviously, when X is σ -compact, we can replace nets by sequences in the above definition and proposition.

Remark 2.4. One can check that if X is a compact space, then we can take a sequence $(f_n)_n$ in $C_c(X \times G)^+$ satisfying $\int_G f_n(x, s) \, ds = 1$ for every $x \in X$ and every $n \in \mathbb{N}$ and such that (2.2) holds.

The following result shows that if X is a σ -compact space, then one can assume that the convergence in (2.1) is uniform on the whole space X , instead of only uniform on compact sets of X .

Proposition 2.5. *Let G be a lsc group, X a σ -compact space and $G \curvearrowright X$ a continuous action. The action $G \curvearrowright X$ is amenable if and only if there exists a sequence of weakly* continuous maps $\mu_n : X \rightarrow \text{Prob}(G)$ satisfying*

$$\lim_{n \rightarrow \infty} \|g \cdot \mu_n(x) - \mu_n(gx)\| = 0$$

uniformly on $x \in X$ and uniformly on compact sets for $g \in G$.

Proof. Suppose that $G \curvearrowright X$ is amenable. Since X is σ -compact, it suffices to construct for every compact set $K \subseteq G$ and every $\varepsilon > 0$ a weakly* continuous map $\mu : X \rightarrow \text{Prob}(G)$ satisfying

$$\|g \cdot \mu(x) - \mu(gx)\| < \varepsilon \tag{2.3}$$

for all $g \in K$ and all $x \in X$.

So, fix a compact set $K \subseteq G$ and an $\varepsilon > 0$. Without loss of generality, we can assume that K is symmetric. Take an increasing sequence $(L_n)_{n \geq 1}$ of compact subsets in X such that $X = \bigcup_n L_n$. Since X is locally compact, after inductively enlarging L_n , we can assume that $L_n \subseteq \text{int}(L_{n+1})$ and $gL_n \subseteq L_{n+1}$ for every $g \in K$. Using the amenability of $G \curvearrowright X$, we can take a sequence of weakly* continuous maps $\nu_n : X \rightarrow \text{Prob}(G)$ satisfying

$$\|g \cdot \nu_n(x) - \nu_n(gx)\| < 2^{-n}$$

for all $g \in K$, $x \in L_n$ and $n \in \mathbb{N} \setminus \{0\}$. Set $L_n = \emptyset$ for $n \leq 0$. Fix $n \geq 1$ such that $18/n < \varepsilon$ and take continuous functions $f_k : X \rightarrow [0, 1]$ such that $f_k(x) = 1$ whenever $x \in L_k \setminus L_{k-n}$ and $f_k(x) = 0$ whenever $x \in L_{k-n-1}$ or $x \in X \setminus L_{k+1}$.

For every $x \in X$, we denote $|x| = \max\{k \in \mathbb{N} \mid x \notin L_k\}$. Set

$$\tilde{\mu}(x) = \sum_{k=0}^{\infty} f_k(x) \nu_k(x) = f_{|x|}(x) \nu_{|x|}(x) + f_{|x|+n+1}(x) \nu_{|x|+n+1}(x) + \sum_{k=|x|+1}^{|x|+n} \nu_k(x).$$

for $x \in X$ and define $\mu : X \rightarrow \text{Prob}(G) : x \mapsto \tilde{\mu}(x) / \|\tilde{\mu}(x)\|$. Clearly, μ is weakly* continuous. To prove that μ satisfies (2.3), fix $x \in X$ and $g \in K$. Since $gL_k \subseteq L_{k+1}$ and $g^{-1}L_k \subseteq L_{k+1}$ for every $k \in \mathbb{N}$, we have $|x| - 1 \leq |gx| \leq |x| + 1$ and hence

$$\|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leq 8 + \sum_{k=|x|+1}^{|x|+n} \|g \cdot \nu_k(x) - \nu_k(gx)\| \leq 9,$$

where we used that $g \in K$ and $x \in L_k$ for $k = |x| + 1, \dots, |x| + n$. Hence,

$$\|g \cdot \mu(x) - \mu(gx)\| \leq \frac{2}{\|\tilde{\mu}(x)\|} \|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leq \frac{18}{n} < \varepsilon$$

as was required. \square

The following result can for instance be found in [BO08, Exercise 15.2.1] in the case of discrete groups. For completeness, we include a proof for locally compact groups here.

Lemma 2.6. *Let G be a lsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, $G \curvearrowright X$ is amenable if and only if the induced action $G \curvearrowright \text{Prob}(X)$ is amenable, where $\text{Prob}(X)$ is equipped with the weak* topology.*

Proof. Since the map $X \rightarrow \text{Prob}(X) : x \mapsto \delta_x$ is weakly* continuous and G -equivariant, we have that amenability of $G \curvearrowright \text{Prob}(X)$ implies amenability of $G \curvearrowright X$.

Conversely, suppose that $G \curvearrowright X$ is amenable. Let $\eta_i : X \rightarrow \text{Prob}(G)$ be a net of maps as in the definition. Then,

$$\tilde{\eta}_i : \text{Prob}(X) \rightarrow \text{Prob}(G) : \mu \mapsto \int_X \eta_i(x) d\mu(x)$$

is weakly* continuous and satisfies

$$\|\tilde{\eta}_i(g \cdot \mu) - g \cdot \tilde{\eta}_i(\mu)\| \leq \int_X \|\eta_i(gx) - g \cdot \eta_i(x)\| d\mu(x) \leq \sup_{x \in X} \|\eta_i(gx) - g \cdot \eta_i(x)\| \rightarrow 0$$

uniformly for $\mu \in \text{Prob}(X)$ and uniformly on compact sets for $g \in G$. \square

2.2 Exactness

The following definition of exactness was given by Kirchberg and Wassermann in [KW99a]. Recall that a G - C^* -algebra is a C^* -algebra A together with a $\|\cdot\|$ -continuous action $G \curvearrowright A$ by $*$ -isomorphisms. We denote by $A \rtimes_r G$ its reduced crossed product.

Definition 2.7. A lcsc group G is called *exact* if for every G -equivariant exact sequence of G - C^* -algebras

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

also the sequence

$$0 \rightarrow A \rtimes_r G \rightarrow B \rtimes_r G \rightarrow C \rtimes_r G \rightarrow 0$$

is exact.

It is an immediate consequence of this definition that the reduced group C^* -algebra $C_r^*(G)$ is exact whenever G is exact. The converse is also true for discrete groups (see [KW99a, Theorem 5.2]), but is still open for locally compact groups. The class of exact groups is very large and contains among others all (weakly) amenable groups [HK94; BCL17], linear groups [GHW05] and hyperbolic groups [Ada94]. By [KW99b, Theorem 4.1 and Theorem 5.1] the class of exact groups is closed under taking closed subgroups and extensions. Examples of non-exact groups were given by Gromov [Gro03; AD08] and Osajda [Osa14].

As before, we denote by $\beta^{lu}G$ the spectrum of the algebra $C_b^{lu}(G)$ of bounded left-uniformly continuous functions on G . The action $G \curvearrowright G$ by left-translation extends uniquely to a continuous action $G \curvearrowright \beta^{lu}G$. By [Ana02, Theorem 7.2] and [BCL17, Theorem A] we have the following equivalent characterizations of exactness.

Theorem 2.8. *Let G be a lcsc group. Then, the following are equivalent.*

- (i) G is exact,
- (ii) G admits a continuous, amenable action on some compact space,
- (iii) the action $G \curvearrowright \beta^{lu}G$ by left-translation is amenable.

3 Class \mathcal{S} and boundary actions small at infinity

The main goal of this section is to prove Theorems B and C, but we first need the following equivalent characterizations of the second condition in the definition of class \mathcal{S} . Note that point (i) in the proposition below is property (S) in the sense of [BDV18].

As before, we denote by $\mathcal{S}(G)$ the space $\{f \in L^1(G)^+ \mid \|f\|_1 = 1\}$ of probability measures on G that are absolutely continuous with respect to the Haar measure. There is an obvious G -equivariant norm-preserving embedding $\mathcal{S}(G) \hookrightarrow \text{Prob}(G)$.

Proposition 3.1. *Let G be a lcsc group. Then, the following are equivalent.*

- (i) There is a $\|\cdot\|_1$ -continuous map $\eta : G \rightarrow \mathcal{S}(G)$ satisfying

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0$$

uniformly on compact sets for $g, h \in G$.

- (ii) There exists a $\|\cdot\|$ -continuous map $\eta : G \rightarrow \text{Prob}(G)$ satisfying

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

- (iii) There exists a Borel map $\eta : G \rightarrow \text{Prob}(G)$ satisfying

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

(iv) There exists a sequence of Borel maps $\eta_n : G \rightarrow M(G)^+$ satisfying

$$\liminf_{n \rightarrow \infty} \liminf_{k \rightarrow \infty} \|\eta_n(k)\| > 0$$

and

$$\lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \sup_{g, h \in K} \|\eta_n(gkh) - g \cdot \eta_n(k)\| = 0$$

for all compact sets $K \subseteq G$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are trivial. We prove the reverse implications (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).

First, we prove (ii) \Rightarrow (i). The proof follows the lines of [Ana02, Proposition 2.2]. Let $\eta : G \rightarrow \text{Prob}(G)$ be as in (ii). We construct $\tilde{\eta} : G \rightarrow \mathcal{S}(G)$ as follows. Take an $f \in C_c(G)^+$ with $\int_G f(t) dt = 1$. Define

$$\tilde{\eta}(g)(s) = \int_G f(t^{-1}s) d\eta(g)(t)$$

for $s, g \in G$. One checks that $\tilde{\eta}(g) \in \mathcal{S}(G)$ for every $g \in G$ and that $\tilde{\eta}$ is $\|\cdot\|_1$ -continuous. For all $g, h, k \in G$ we have

$$\begin{aligned} \int_G |\tilde{\eta}(gkh)(s) - \tilde{\eta}(k)(g^{-1}s)| ds &= \int_G \left| \int_G f(t^{-1}s) d\eta(gkh)(t) - \int_G f(t^{-1}g^{-1}s) d\eta(k)(t) \right| ds \\ &\leq \int_G \int_G f(t^{-1}s) d|\eta(gkh) - g \cdot \eta(k)|(t) ds \\ &= \|\eta(gkh) - g \cdot \eta(k)\| \end{aligned}$$

which tends to zero uniformly on compact sets for $g, h \in G$ whenever $k \rightarrow \infty$.

The implication (iii) \Rightarrow (ii) follows immediately by applying Lemma 3.4 below with $H = Y = G$ and the actions $G \times G \curvearrowright H$ and $G \curvearrowright Y$ defined by $(g, k) \cdot h = ghk^{-1}$ and $(g, h) \cdot y = gy$ for $g, k \in G$, $h \in H$ and $y \in Y$.

Finally, (iv) \Rightarrow (iii) follows from the technical lemma 3.2 below applied on the spaces $X = Y = G$ with the same actions as above. \square

The following is a more abstract and slightly more general version of the trick in [BO08, Exercise 15.1.1]. It will be used several times in this article.

Lemma 3.2. *Let X and Y be σ -compact spaces and G a lcsc group. Suppose that $G \curvearrowright X$ and $G \curvearrowright Y$ are continuous actions. If there exists a sequence of Borel maps $\eta_n : X \rightarrow M(Y)^+$ satisfying*

$$\lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \sup_{g \in K} \|\eta_n(gx) - g \cdot \eta_n(x)\| = 0$$

for all compact sets $K \subseteq G$ and

$$\liminf_{n \rightarrow \infty} \liminf_{x \rightarrow \infty} \|\eta_n(x)\| > 0.$$

Then, there exists a Borel map $\eta : X \rightarrow \text{Prob}(Y)$ such that

$$\lim_{x \rightarrow \infty} \|\eta(gx) - g \cdot \eta(x)\| = 0 \tag{3.1}$$

uniformly on compact sets for $g \in G$. Moreover, if the maps η_n are assumed to be $\|\cdot\|$ -continuous, then also η can be assumed to be $\|\cdot\|$ -continuous. If the maps η_n are weakly* continuous and the maps $x \mapsto \|\eta_n(x)\|$ are continuous for every $n \in \mathbb{N}$, then also η can be assumed to be weakly* continuous.

Proof. After passing to a subsequence and replacing values of η_n in a compact set, we can assume that there exists a $\delta > 0$ such that $\|\eta_n(x)\| \geq \delta$ for all $n \in \mathbb{N}$ and all $x \in X$. Set $\tilde{\eta}_n(x) = \eta_n(x) / \|\eta_n(x)\|$ for all $x \in X$. Note that

$$\lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \sup_{g \in K} \|\tilde{\eta}_n(gx) - g \cdot \tilde{\eta}_n(x)\| \leq \lim_{n \rightarrow \infty} \limsup_{x \rightarrow \infty} \sup_{g \in K} \frac{2}{\|\eta_n(x)\|} \|\eta_n(gx) - g \cdot \eta_n(x)\| = 0$$

for all compact sets $K \subseteq G$.

Take an increasing sequence $(K_n)_n$ of compact symmetric neighborhoods of the unit e in G such that $G = \bigcup_n \text{int}(K_n)$. After passing to a subsequence of $(\tilde{\eta}_n)_n$, we find compact sets $L_n \subseteq X$ such that

$$\|\tilde{\eta}_n(gx) - g \cdot \tilde{\eta}_n(x)\| \leq 2^{-n+1}$$

for all $g \in K_n$ and $x \in X \setminus L_n$. After inductively enlarging L_n , we can assume that the sequence $(L_n)_n$ is increasing, that $gL_n \subseteq L_{n+1}$ for all $g \in K_n$ and that $X = \bigcup_n L_n$. Moreover, we can also assume L_0 to be the empty set.

For every $x \in X$, we denote $|x| = \max\{n \in \mathbb{N} \mid x \notin L_n\}$. Furthermore, we denote $h(n) = \lfloor n/2 \rfloor + 1$ for $n \geq 1$. Fix a $y_0 \in Y$. We set $\mu(x) = \delta_{y_0}$ whenever $|x| \leq 1$ and

$$\mu(x) = \sum_{k=h(|x|)}^{|x|} \tilde{\eta}_k(x)$$

whenever $|x| \geq 2$. Now, define $\eta : X \rightarrow \text{Prob}(Y)$ by $\eta(x) = \mu(x) / \|\mu(x)\|$.

To prove that η satisfies (3.1), take $\varepsilon > 0$ and $K \subseteq G$ arbitrary. Since $\bigcup_n \text{int}(K_n) = G$, we can take an $n_0 \geq 1$ such that $K \subseteq K_{n_0}$. Take $n_1 > \max\{2n_0, 16/\varepsilon\}$. We claim that $\|\eta(g \cdot x) - g \cdot \eta(x)\| < \varepsilon$ whenever $x \in G \setminus L_{n_1}$ and $g \in K$. Indeed, fix $g \in K$ and $x \in G \setminus L_{n_1}$. Take $n \geq n_1$ such that $x \in L_{n+1} \setminus L_n$. Then, $|x| = n$. Since $gL_{n+1} \subseteq L_{n+2}$ and $g^{-1}L_{n-1} \subseteq L_n$, we have that $gx \in L_{n+2} \setminus L_{n-1}$ and hence $n-1 \leq |gx| \leq n+1$. This yields

$$\|\mu(gx) - g \cdot \mu(x)\| \leq 2 + \sum_{k=h(n)}^n \|\tilde{\eta}_k(gx) - g \cdot \tilde{\eta}_k(x)\| \leq 4,$$

since $g \in K_k$ and $x \in X \setminus L_k$ for $k = h(n), \dots, n$. Hence

$$\|\eta(gx) - g \cdot \eta(x)\| \leq \frac{2}{\|\mu(x)\|} \|\mu(gx) - g \cdot \mu(x)\| \leq \frac{4}{n} \cdot 4 < \varepsilon$$

which proves the claim.

If the maps η_n are $\|\cdot\|$ -continuous (resp. weakly* continuous and such that $x \mapsto \|\eta_n(x)\|$ is continuous), then we can make μ (and hence η) $\|\cdot\|$ -continuous (resp. weakly* continuous) in the following way. Note first that then $\tilde{\eta}$ is also $\|\cdot\|$ -continuous (resp. weakly* continuous). By inductively enlarging the compact sets L_n above and using that X is locally compact, we can assume that $L_n \subseteq \text{int}(L_{n+1})$. For all $n \geq 1$, we take a continuous function $f_n : X \rightarrow [0, 1]$ such that $f_n(x) = 1$ if $x \in L_{2n} \setminus L_n$ and $f_n(x) = 0$ if $x \in L_{n-1}$ or $x \in X \setminus L_{2n+1}$. For $x \in X$ with $|x| \geq 2$, we set

$$\tilde{\mu}(x) = \sum_{k=1}^{+\infty} f_k(x) \tilde{\eta}_k(x) = f_{h(|x|)-1}(x) \tilde{\eta}_{h(|x|)-1}(x) + f_{|x|+1}(x) \tilde{\eta}_{|x|+1}(x) + \sum_{k=h(|x|)}^{|x|} \tilde{\eta}_k(x).$$

Fix again $y_0 \in Y$ and take a continuous map $a : X \rightarrow [0, 1]$ such that $a(x) = 1$ if $x \in L_2$ and $a(x) = 0$ when $x \in X \setminus L_3$. We define the continuous map $\eta : X \rightarrow M(Y)^+$ by

$$\mu(x) = \begin{cases} a(x)\delta_{y_0} + (1-a(x)) \frac{\tilde{\mu}(x)}{\|\tilde{\mu}(x)\|} & \text{if } |x| \geq 2 \\ \delta_{y_0} & \text{if } |x| \leq 1 \end{cases}$$

Obviously, η is $\|\cdot\|$ -continuous (resp. weakly* continuous) and, by a similar calculation as above, one proves that η satisfies (3.1). \square

Remark 3.3. Using almost exactly the same proof as above, one can actually prove the following slightly more general result: suppose that for every $\varepsilon > 0$, every compact set $K \subseteq G$, there exists a compact set $L \subseteq X$ such that for all compact sets $L' \subseteq X$, there exists a map $\mu : X \rightarrow M(Y)^+$ such that

$$\frac{\|\mu(gx) - g \cdot \mu(x)\|}{\|\mu(x)\|} < \varepsilon \tag{3.2}$$

whenever $g \in K$ and $x \in L' \setminus L$. Then, there exists a map $\eta : X \rightarrow \text{Prob}(Y)$ as in (3.1). Indeed, using the notation of the proof, we can take the compact sets $L_n \subseteq X$ and the maps $\eta_n : X \rightarrow M(Y)^+$ such that

$$\|\tilde{\eta}_n(gx) - g \cdot \tilde{\eta}_n(x)\| \leq \frac{2}{\|\eta_n(x)\|} \|\eta_n(gx) - g \cdot \eta_n(x)\| < 2^{-n+1}$$

for all $g \in K_n$ and $x \in L_{2n} \setminus L_n$, where again $\tilde{\eta}_n(x) = \eta_n(x) / \|\eta_n(x)\|$. The rest of the proof holds verbatim.

The following lemma will be used several times to replace Borel maps by continuous maps.

Lemma 3.4. *Let H and G be lsc groups and Y a locally compact space. Suppose that $G \curvearrowright^\alpha H$ and $G \curvearrowright Y$ are arbitrary continuous actions. If there exists a Borel map $\eta : H \rightarrow \text{Prob}(Y)$ satisfying*

$$\lim_{h \rightarrow \infty} \|\eta(\alpha_g(h)) - g \cdot \eta(h)\| = 0 \quad \text{and} \quad \lim_{h \rightarrow \infty} \|\eta(\alpha_g(h)k) - \eta(\alpha_g(hk))\| = 0$$

uniformly on compact sets for $g \in G$ and $k \in H$, then there exists a $\|\cdot\|$ -continuous map $\tilde{\eta} : H \rightarrow \text{Prob}(Y)$ map satisfying

$$\lim_{h \rightarrow \infty} \|\tilde{\eta}(\alpha_g(h)) - g \cdot \tilde{\eta}(h)\| = 0$$

Proof. Fix a compact neighborhood K of the unit e in H with $\lambda_H(K) = 1$. We define $\tilde{\eta} : H \rightarrow \text{Prob}(Y)$ by

$$\tilde{\eta}(g) = \int_K \eta(gk) dk.$$

The map $\tilde{\eta}$ is continuous, since for $h_1, h_2 \in H$ we have

$$\|\tilde{\eta}(h_1) - \tilde{\eta}(h_2)\| \leq \int_{h_1 K \Delta h_2 K} \|\eta(k)\| dk = \lambda_H(h_1 K \Delta h_2 K)$$

and the right hand side tends to zero whenever $h_2 \rightarrow h_1$. Moreover, for $g \in G$ and $h \in H$, we have

$$\|\tilde{\eta}(\alpha_g(h)) - g \cdot \tilde{\eta}(h)\| \leq \int_K \|\eta(\alpha_g(h)k) - g \cdot \eta(hk)\| dk$$

Since K is compact the right hand side tends to zero uniformly on compact sets for $g \in G$ whenever $h \rightarrow \infty$. \square

We are now ready to prove Theorem B.

Proof of Theorem B. First, we prove (i) \Rightarrow (ii). Let $\eta : G \rightarrow \text{Prob}(G)$ be a map as in the definition of class \mathcal{S} . Consider the u.c.p. map $\eta_* : C_b^{lu}(G) \rightarrow C_b^u(G)$ defined by

$$(\eta_* f)(g) = \int_G f(s) d\eta(g)(s)$$

for $f \in C_b^{lu}(G)$ and $g \in G$. Note that η_* is well-defined. Indeed, fix $f \in C_b^{lu}(G)$ and $\varepsilon > 0$. For $g, h \in G$ we have

$$|(\eta_* f)(h^{-1}g) - (\eta_* f)(g)| \leq \min \{ \|f\|_\infty \|\eta(h^{-1}g) - \eta(g)\|, \|f\|_\infty \|\eta(h^{-1}g) - h^{-1} \cdot \eta(g)\| + \|f - \lambda_h f\|_\infty \}$$

and

$$|(\eta_* f)(gh) - (\eta_* f)(g)| \leq \|f\|_\infty \|\eta(gh) - \eta(g)\|. \quad (3.3)$$

Pick a compact neighborhood K of the unit e in G . We find a compact subset $L \subseteq G$ such that

$$\|\eta(gh) - \eta(g)\| \leq \varepsilon \quad \text{and} \quad \|\eta(h^{-1}g) - h^{-1} \cdot \eta(g)\| \leq \frac{\varepsilon}{2}$$

whenever $h \in K$ and $g \in G \setminus L$. Now, we can take an open neighborhood $\mathcal{U} \subseteq K$ of the unit e in G such that

$$\|f - \lambda_h f\|_\infty \leq \frac{\varepsilon}{2}, \quad \sup_{g \in L} \|\eta(h^{-1}g) - \eta(g)\| \leq \varepsilon \quad \text{and} \quad \sup_{g \in L} \|\eta(gh) - \eta(h)\| \leq \varepsilon$$

for all $h \in \mathcal{U}$. It follows that

$$\|\lambda_h(\eta_* f) - \eta_* f\|_\infty \leq \varepsilon \quad \text{and} \quad \|\rho_h(\eta_* f) - \eta_* f\|_\infty < \varepsilon$$

for all $h \in \mathcal{U}$.

Moreover, (3.3) also implies that

$$\lim_{g \rightarrow \infty} |(\eta_* f)(gh) - (\eta_* f)(g)| = 0$$

for all $h \in G$ and hence that $\eta_*(f) \in C(h^u G)$ for all $f \in C_b^{lu}(G)$.

Similarly, one proves that $\eta_*(\lambda_g f) - \lambda_g(\eta_* f) \in C_0(G)$. Let $\pi : C(h^u G) \rightarrow C(\nu^u G) \cong C(h^u G)/C_0(G)$ be the quotient map. It follows that $\pi \circ \eta_* : C_b^{lu}(G) \rightarrow C(\nu^u G)$ is a G -equivariant u.c.p. map. By dualization, we obtain a weakly* continuous G -equivariant map $\nu^u G \rightarrow \text{Prob}(\beta^{lu} G)$ given by $x \mapsto \delta_x \circ \pi \circ \eta_*$. Since G is exact, the action $G \curvearrowright \beta^{lu} G$ is amenable and hence so is $G \curvearrowright \text{Prob}(\beta^{lu} G)$ (see Lemma 2.6). It follows that $G \curvearrowright \nu^u G$.

Now, we prove (ii) \Leftrightarrow (iii). The implication from right to left is trivial. To prove the other implication, take an arbitrary compact subset $K \subseteq G$ and an $\varepsilon > 0$. By Proposition 2.2, it suffices to construct a function $h \in C_c(h^u G \times G)^+$ such that $\int_G h(x, s) ds = 1$ for every $x \in h^u G$ and

$$\int_G |h(x, g^{-1}s) - h(gx, s)| ds < \varepsilon \quad (3.4)$$

for all $x \in h^u G$ and $g \in K$.

By Proposition 2.2 and Remark 2.4, we find an $f \in C_c(\nu^u G \times G)^+$ satisfying $\int_G f(x, s) ds = 1$ and

$$\int_G |f(x, g^{-1}s) - f(gx, s)| ds < \frac{\varepsilon}{2}$$

for all $x \in \nu^u G$ and $g \in K$. Denote

$$S = \sup_{g \in K} \sup_{x \in \nu^u G} \int_G |f(x, g^{-1}s) - f(gx, s)| ds < \frac{\varepsilon}{2}$$

and let $\delta = \varepsilon/2 - S$. By the Tietze Extension Theorem, we can extend f to a function $\tilde{f} \in C_c(h^u G \times G)^+$. Since

$$\limsup_i \int_G |\tilde{f}(x_i, g_i^{-1}s) - \tilde{f}(g_i x_i, s)| ds \leq S,$$

for every net $(x_i)_i$ in G converging to an $x \in \nu^u G$ and every net $(g_i)_i$ in K , we can take a compact set $L \subseteq G$ such that

$$\int_G |\tilde{f}(x, g^{-1}s) - \tilde{f}(gx, s)| ds < S + \frac{\delta}{2}.$$

for all $x \in h^u G \setminus L$ and $g \in K$. Since $\lim_i \int_G \tilde{f}(x_i, s) ds = 1$ for every net $(x_i)_i$ in G converging to an $x \in \nu^u G$, we can after enlarging L also assume that

$$\left| \int_G \tilde{f}(x, s) ds - 1 \right| \leq \frac{\delta}{4} \quad \text{and} \quad \left| \int_G \tilde{f}(gx, s) ds - 1 \right| \leq \frac{\delta}{4}$$

for $x \in G \setminus L$ and $g \in K$. Renormalizing \tilde{f} then yields a map satisfying $\int_G \tilde{f}(x, s) ds = 1$ and

$$\int_G |\tilde{f}(x, g^{-1}s) - \tilde{f}(gx, s)| ds < \frac{\varepsilon}{2}.$$

Now, fix a function $a \in C_c(G)^+$ with $\int_G a(s) ds = 1$. Using Lemma 3.5 below, we take a function $\zeta \in C_c(G)^+$ such that $\zeta|_L = 1$ and $|\zeta(gh) - \zeta(h)| < \varepsilon/4$ for $h \in G$ and $g \in K$. Now, define $h \in C_c(h^u G \times G)$ by

$$h(x, s) = \begin{cases} \zeta(x)a(x^{-1}s) + (1 - \zeta(x))\tilde{f}(x, s) & \text{if } x \in G, \\ \tilde{f}(x, s) & \text{if } x \in \nu^u G. \end{cases}$$

A straightforward calculation shows that h satisfies (3.4).

Next, we prove (ii) \Rightarrow (iv). Denote by X the spectrum of $A = C_b^u(G)/C_0(G)$. Since $C(h^u G) \subseteq C_b^u(G)$, we have a natural embedding $C(\nu^u G) \hookrightarrow A$, which in turn induces a continuous map $\varphi_\ell : X \rightarrow \nu^u G$. Note that φ_ℓ is $G \times G$ -equivariant with respect to the actions induced by left and right translation. Similarly, we get a $G \times G$ -equivariant map $\varphi_r : X \rightarrow \nu_r^u G$, where $\nu_r^u G$ denotes the spectrum of the algebra

$$C(\nu_r^u G) = \{f \in C_b^u(G) \mid \lambda_g f - f \in C_0(G)\}$$

and the action $G \times G \curvearrowright \nu_r^u G$ is induced by left and right translation. By assumption, the action $G \times 1 \curvearrowright \nu^u G$ is amenable, and by symmetry so is $1 \times G \curvearrowright \nu_r^u G$. Since the actions $1 \times G \curvearrowright \nu^u G$ and $G \times 1 \curvearrowright \nu_r^u G$ are trivial, the

diagonal action $G \times G \curvearrowright \nu^u G \times \nu_r^u G$ is amenable. Now, the conclusion follows from the $G \times G$ -equivariance of the map $\varphi_\ell \times \varphi_r : X \rightarrow \nu^u G \times \nu_r^u G$.

Finally, we prove (iv) \Rightarrow (i). By Theorem 2.8, the group G is exact. Denote again by X the spectrum of $A = C_b^u(G)/C_0(G)$. Denoting by $\beta^u G$ the spectrum of $C_b^u(G)$, we get $X = \beta^u G \setminus G$. By Proposition 2.2 and Remark 2.4, we can take a sequence $(f_n)_n$ of functions in $C_c(X \times G \times G)^+$ such that $\int_{G \times G} f_n(x, s, t) ds dt = 1$ for all $x \in X$ and $n \in \mathbb{N}$, and such that

$$\lim_{n \rightarrow \infty} \int_{G \times G} |f_n(x, g^{-1}s, h^{-1}t) - f_n((g, h) \cdot x, s, t)| ds dt = 0 \quad (3.5)$$

uniformly for $x \in X$ and uniformly on compact sets for $g, h \in G$. As before, the Tietze Extension Theorem yields extensions $\tilde{f}_n \in C_c(\beta^u G \times G \times G)^+$ of each f_n . For each $x \in \beta^u G$ and $n \in \mathbb{N}$, we define $\eta_n(x) \in M(G)^+$ as the measure with density function

$$s \mapsto \int_G \tilde{f}_n(x, s, t) dt.$$

with respect to the Haar measure. This yields $\|\cdot\|$ -continuous maps $\eta_n : \beta^u G \rightarrow M(G)^+$. By (3.5), the restrictions of η_n to $G \subseteq \beta^u G$ satisfy the conditions of Proposition 3.1 (iv). \square

In the proof above, we used the following easy lemma.

Lemma 3.5. *Let G be a lcsc group. For all compact subsets $K, L \subseteq G$ and all $\varepsilon > 0$, there exists a continuous function $f \in C_c(G)$ satisfying $f|_L = 1$ and*

$$|f(kgk') - f(g)| < \varepsilon$$

for $k, k' \in K$ and $g \in G$.

Proof. Without loss of generality we can assume that K is symmetric and that $\text{int}(K)$ contains the unit e . Denote $L_0 = L$ and $L_n = K^n L K^n$ for $n \geq 1$. Then, $L_n \subseteq \text{int}(L_{n+1})$ for every $n \in \mathbb{N}$ and hence, we can take a continuous $f_n : G \rightarrow [0, 1]$ with $f_n(g) = 1$ for $g \in L_n$ and $\text{supp } f_n \subseteq L_{n+1}$. Take $N \in \mathbb{N}$ such that $1/N < \varepsilon/4$ and set

$$f(g) = \frac{1}{N} \sum_{k=0}^{N-1} f_k(g).$$

Then, f satisfies the conclusions of the lemma. \square

We end this section by proving Theorem C.

Proof of Theorem C. The implication from left to right is trivial. To prove the converse implication, note that by exactness of G and Lemma 2.6, the action $G \curvearrowright \text{Prob}(\beta^{lu} G)$ is amenable. Take a sequence $\theta_n : \text{Prob}(\beta^{lu} G) \rightarrow \text{Prob}(G)$ such that

$$\lim_{n \rightarrow \infty} \|\theta_n(g \cdot \mu) - g \cdot \theta_n(\mu)\| = 0$$

uniformly for $\mu \in \text{Prob}(\beta^{lu} G)$ and uniformly on compact sets for $g \in G$. Now, for the composition $\eta_n = \theta_n \circ \eta$ we get

$$\begin{aligned} \|\eta_n(gkh) - g \cdot \eta_n(k)\| &\leq \|\eta(gkh) - g \cdot \eta(k)\| + \|\theta_n(g \cdot \eta(k)) - g \cdot \theta_n(\eta(k))\| \\ &\leq \|\eta(gkh) - g \cdot \eta(k)\| + \sup_{\mu \in \text{Prob}(\beta^{lu} G)} \|\theta_n(g \cdot \mu) - g \cdot \theta_n(\mu)\| \end{aligned}$$

whenever $g, h, k \in G$. It follows that

$$\lim_{n \rightarrow \infty} \limsup_{k \rightarrow \infty} \sup_{g, h \in K} \|\eta_n(gkh) - g \cdot \eta_n(k)\| = 0$$

for every compact set $K \subseteq G$. Hence, Lemma 3.2 concludes the proof. \square

4 Locally compact wreath products in class \mathcal{S}

In this section, we prove Theorem D. Before we can start the proof of this result, we need a few lemmas. The first lemma is a locally compact version of [BO08, Lemma 15.2.6]. This result can be proven in a similar way as in [BO08, Lemma 15.2.6]. However, we provide a different proof not requiring exactness.

Proposition 4.1. *Let G be an lsc group and K a closed, amenable subgroup. If there exists a Borel map $\eta : G \rightarrow \text{Prob}(G/K)$ such that*

$$\lim_{k \rightarrow \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$. Then, G has property (S), i.e. there exists a $\|\cdot\|$ -continuous map $\tilde{\eta} : G \rightarrow \text{Prob}(G)$ satisfying (1.1).

Proof. Using Lemma 3.4 we can assume that η is $\|\cdot\|$ -continuous. The proof then follows easily from Lemma 4.3 below. \square

Let G be a group and $H \subseteq G$ a closed subgroup. Denote by $p : G \rightarrow G/H$ the quotient map. Let $\sigma : G/H \rightarrow G$ be a locally bounded Borel cross section for p , i.e. a Borel map satisfying $p \circ \sigma = \text{Id}_{G/H}$ that maps compact sets onto precompact sets (see for instance [Mac52, Lemma 1.1] for the existence of such a map). We can identify G with $G/H \times H$ via the map the map

$$\phi : G \rightarrow G/H \times H : g \mapsto (gH, \sigma(gH)^{-1}g). \quad (4.1)$$

Under this identification the action by left translation is given by $k \cdot (gH, h) = (kgH, \omega(k, gH)h)$, where $\omega(k, gH) = \sigma(kgH)^{-1}k\sigma(gH)$. Note that ω maps compact sets of $G \times G/H$ onto precompact sets of G .

The identification map ϕ is not continuous, but it is bi-measurable and maps (pre)compact sets to precompact sets. This allows us to identify the spaces $\text{Prob}(G)$ and $\text{Prob}(G/H \times H)$ via the map $\mu \mapsto \phi_*\mu$. Note that this identification map is continuous with respect to the norm topology on both spaces (and hence bi-measurable), but not with respect to the weak* topology on both spaces. We use the above identifications in the following two lemmas.

Lemma 4.2. *Let G be a lsc group and $H \subseteq G$ a closed, amenable subgroup. Let $(\nu_n)_n$ be a sequence in $\text{Prob}(H)$ satisfying*

$$\lim_{n \rightarrow \infty} \|h \cdot \nu_n - \nu_n\| = 0$$

uniformly on compact sets for $h \in H$. Then,

$$\lim_{n \rightarrow \infty} \|h \cdot (\mu \otimes \nu_n) - (h \cdot \mu) \otimes \nu_n\| = 0$$

uniformly on compact sets for $g \in G$ and $\mu \in \text{Prob}(G/H)$, where we equipped $\text{Prob}(G/H)$ with the weak topology.*

Proof. Fix compact subsets $K \subseteq G$ and $\mathcal{L} \subseteq \text{Prob}(G/H)$. Take an arbitrary $\varepsilon > 0$. There is a compact subset $L \subseteq G/H$ such that $\mu(L) > 1 - \varepsilon$ for all $\mu \in \mathcal{L}$. Hence, for all $f \in C_c(G/H \times H)$, $\mu \in \mathcal{L}$, $k \in G$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} & \left| \int_{G/H \times H} f dk \cdot (\mu \otimes \nu_n) - \int_{G/H \times H} f d(k \cdot \mu) \otimes \nu_n \right| \\ &= \left| \int_{G/H} \int_H f(k \cdot (gH, h)) d\nu_n(h) d\mu(gH) - \int_{G/H} \int_H f(kgH, h) d\nu_n(h) d\mu(gH) \right| \\ &= \left| \int_{G/H} \int_H f(kgH, \omega(k, gH)h) d\nu_n(h) d\mu(gH) - \int_{G/H} \int_H f(kgH, h) d\nu_n(h) d\mu(gH) \right| \\ &\leq \int_{G/H} \int_H |f(kgH, h)| d|\omega(k, gH) \cdot \nu_n - \nu_n|(h) d\mu(gH) \\ &\leq \|f\|_\infty \left(2\varepsilon + \int_L \|\omega(k, gH) \cdot \nu_n - \nu_n\| d\mu(gH) \right), \end{aligned}$$

where $|\omega(k, gH) \cdot \nu_n - \nu_n|$ denotes the total variation measure of $\omega(k, gH) \cdot \nu_n - \nu_n$. Since ω maps compact sets to precompact sets, we can find an $n_0 \in \mathbb{N}$ such that $\|\omega(k, gH) \cdot \nu_n - \nu_n\| < \varepsilon$ for all $n \geq n_0$, all $k \in K$ and all $gH \in L$. We conclude that

$$\|k \cdot (\mu \otimes \nu_n) - (k \cdot \mu) \otimes \nu_n\| \leq 3\varepsilon$$

whenever $n \geq n_0$, $\mu \in \mathcal{L}$ and $k \in K$, thus proving the result. \square

Lemma 4.3. *Let G and H be lcsc groups, $\pi : G \rightarrow H$ a continuous morphism and $K \subseteq H$ a closed, amenable subgroup. Let $G \curvearrowright X$ be a continuous action on some σ -compact space X . Let $G \curvearrowright \text{Prob}(H)$ (resp. $G \curvearrowright \text{Prob}(H/K)$) be defined by $g \cdot \mu = \pi(g) \cdot \mu$ for $g \in G$ and $\mu \in \text{Prob}(H)$ (resp. $\mu \in \text{Prob}(H/K)$). If there exists a weakly* continuous map $\eta : X \rightarrow \text{Prob}(H/K)$ such that*

$$\lim_{x \rightarrow \infty} \|\eta(gx) - g \cdot \eta(x)\| = 0$$

uniformly on compact sets for $g \in G$. Then, there exists a Borel map $\tilde{\eta} : X \rightarrow \text{Prob}(H)$ such that

$$\lim_{x \rightarrow \infty} \|\tilde{\eta}(gx) - g \cdot \tilde{\eta}(x)\| = 0$$

uniformly on compact sets for $g \in G$. Moreover, if η is assumed to be $\|\cdot\|$ -continuous then also $\tilde{\eta}$ can be assumed to be $\|\cdot\|$ -continuous.

Proof. Fixing a locally bounded Borel cross section $\sigma : H/K \rightarrow H$ for the quotient map $p : H \rightarrow H/K$, we can identify H with $H/K \times K$ and $\text{Prob}(H)$ with $\text{Prob}(H/K \times K)$ as in (4.1).

Since K is amenable, we can take a sequence $(\nu_n)_n$ in $\text{Prob}(K)$ such that $\|k \cdot \nu_n - \nu_n\| \rightarrow 0$ uniformly on compact sets for $k \in K$ whenever $n \rightarrow \infty$. Using Lemma 4.2, we construct maps as in Remark 3.3 as follows. Fix an $\varepsilon > 0$ and a compact $C \subseteq G$. Take a compact $L \subseteq X$ such that $\|\eta(gx) - g \cdot \eta(x)\| < \varepsilon$ for all $g \in C$ and $x \in X \setminus L$. Fix any compact set $L' \subseteq X$. Applying Lemma 4.2 to the weak* compact set $\eta(L')$, we find an $n \in \mathbb{N}$ such that

$$\|(g \cdot \eta(x)) \otimes \nu_n - g \cdot (\eta(x) \otimes \nu_n)\| < \varepsilon$$

for any $x \in L'$ and $g \in C$. Hence,

$$\|\eta(gx) \otimes \nu_n - g \cdot (\eta(x) \otimes \nu_n)\| \leq \|\eta(gx) - g \cdot \eta(x)\| + \|(g \cdot \eta(x)) \otimes \nu_n - g \cdot (\eta(x) \otimes \nu_n)\| \leq 2\varepsilon$$

for any $g \in C$ and any $x \in L' \setminus L$. We conclude that the map $\mu : X \rightarrow \text{Prob}(H)$ defined by $\mu(x) = \eta(x) \otimes \nu_n$ is as in (3.2). Moreover, if η is $\|\cdot\|$ -continuous, then so is μ . \square

The second result that we need before proving Theorem D characterizes when a semi-direct product belongs to class \mathcal{S} . By definition a semi-direct product $G = B \rtimes H$ belongs to class \mathcal{S} whenever it is exact and there exists a map $\eta : G \rightarrow \text{Prob}(G)$ satisfying

$$\|\mu((a, k)(b, h)(a', k')) - (a, k) \cdot \mu(b, h)\| \rightarrow 0$$

uniformly on compact sets for $(a, k), (a', k') \in G$ whenever $(b, h) \rightarrow \infty$. The result below shows that it suffices that there exist *two* such maps one of which satisfies the convergence above when $b \rightarrow \infty$ and the other when $h \rightarrow \infty$.

Proposition 4.4. *Let $G = B \rtimes_{\alpha} H$ be a semi-direct product of lcsc groups. Then, G is in class \mathcal{S} if and only if B and H are exact, and there exists Borel maps $\mu : G \rightarrow \text{Prob}(G)$ and $\nu : G \rightarrow \text{Prob}(G)$ such that*

$$\lim_{b \rightarrow \infty} \|\mu((a, k)(b, h)(a', k')) - (a, k) \cdot \mu(b, h)\| = 0 \tag{4.2}$$

uniformly on compact sets for $a, a' \in B$ and $k, h, k' \in H$, and such that

$$\lim_{h \rightarrow \infty} \|\nu((a, k)(b, h)(a', k')) - (a, k) \cdot \nu(b, k)\| \tag{4.3}$$

uniformly for $b \in B$ and uniformly on compact sets for $a, a' \in B$ and $k, k' \in H$.

Proof. The only if part is immediate. Indeed, the map $\eta : G \rightarrow \text{Prob}(G)$ as in the definition of class \mathcal{S} , satisfies both (4.2) and (4.3). Moreover, the groups B and H are exact as subgroups of an exact group.

To prove the converse, note first that G is exact as an extension of an exact group by an exact group (see [KW99b, Theorem 5.1]). Let $\mu : G \rightarrow \text{Prob}(G)$ and $\nu : G \rightarrow \text{Prob}(G)$ be as above. By Proposition 3.1, it suffices to prove that for every compact $K \subseteq G$ and every $\varepsilon > 0$, there exists a Borel map $\eta : G \rightarrow \text{Prob}(G)$ and a compact $L \subseteq G$ such that

$$\|\eta((a, k)(b, h)(a', k')) - (a, k) \cdot \eta(b, h)\| < \varepsilon \tag{4.4}$$

for all $(a, k), (a', k') \in K$ and all $(b, h) \in G \setminus L$.

So, fix a compact $K \subseteq G$ and an $\varepsilon > 0$. Let $K_B \subseteq B$ and $K_H \subseteq H$ be compact subsets such that $K \subseteq \{(b, h) \mid b \in K_B, h \in K_H\}$. By assumption, we can take a compact set $\tilde{L}_H \subseteq H$ such that

$$\|\nu((a, k)(b, h)(a', k')) - (a, k) \cdot \nu(b, h)\| < \frac{\varepsilon}{2} \quad (4.5)$$

whenever $a, a' \in K_B, b \in B, k, k' \in K_H$ and $h \in H \setminus \tilde{L}_H$.

Using Lemma 3.5, we take a function $f \in C_c(H)$ such that $f(h) = 1$ for $h \in \tilde{L}_H$ and $|f(khk') - f(h)| < \varepsilon/4$ whenever $h \in H$ and $k, k' \in K_H$. Set $L_H = \text{supp } f$. Now, we can take a compact set $L_B \subseteq B$ such that

$$\|\mu((a, k)(b, h)(a', k')) - (a, k) \cdot \mu(b, h)\| < \frac{\varepsilon}{2} \quad (4.6)$$

whenever $a, a' \in K_B, b \in G \setminus L_B, k, k' \in K_H$ and $h \in L_H$.

Define $\eta : G \rightarrow \text{Prob}(G)$ by

$$\eta(b, h) = f(h)\mu(b, h) + (1 - f(h))\nu(b, h)$$

for $(b, h) \in G$. Set $L = \{(b, h) \in G \mid b \in L_B, h \in L_H\}$. Fix $(a, k), (a', k') \in K$ and $(b, h) \in G \setminus L$. We have

$$\begin{aligned} \|\eta((a, k)(b, h)(a', k')) - (a, k) \cdot \eta(b, h)\| &\leq f(h) \|\mu((a, k)(b, h)(a', k')) - (a, k) \cdot \mu(b, h)\| \\ &\quad + (1 - f(h)) \|\nu((a, k)(b, h)(a', k')) - (a, k) \cdot \nu(b, h)\| + 2|f(khk') - f(h)| \\ &\leq f(h) \|\mu((a, k)(b, h)(a', k')) - (a, k) \cdot \mu(b, h)\| \\ &\quad + (1 - f(h)) \|\nu((a, k)(b, h)(a', k')) - (a, k) \cdot \nu(b, h)\| + \frac{\varepsilon}{2} \end{aligned}$$

We are in one of the following three cases.

Case 1. If $h \in H \setminus L_H$, then $f(h) = 0$ and (4.5) holds.

Case 2. If $h \in L_H \setminus \tilde{L}_H$ and $b \in B \setminus L_B$, then both (4.5) and (4.6) hold.

Case 3. If $h \in \tilde{L}_H$ and $b \in B \setminus L_B$, then $f(h) = 1$ and (4.6) holds.

In all three cases, we conclude that (4.4) holds, thus proving the proposition. \square

Remark 4.5. Note that (4.2) is equivalent with the existence of a map $\tilde{\mu} : B \rightarrow \text{Prob}(G)$ satisfying

$$\lim_{b \rightarrow \infty} \|\tilde{\mu}(aba') - a \cdot \tilde{\mu}(b)\| = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} \|\tilde{\mu}(\alpha_h(b)) - h \cdot \tilde{\mu}(b)\| = 0$$

uniformly on compact sets for $a, a' \in B$ and $h \in H$. Indeed, the restriction of a map as in (4.2) satisfies the above equations. Conversely, given a map $\tilde{\mu}$ as above, the map $\mu : G \rightarrow \text{Prob}(G)$ defined by $\mu(b, h) = \tilde{\mu}(b)$ satisfies (4.2).

When the group B is amenable, the previous result specializes to the corollary below. In the setting of countable groups, this result was proved by Ozawa in [Oza06, proof of Corollary 4.5] and [Oza09, Section 3]. However, the proof provided there does not carry over to the locally compact setting, since, as we explained in the introduction, the characterization of class \mathcal{S} in terms of a u.c.p. map $\varphi : C_r^*(G) \otimes_{\min} C_r^*(G) \rightarrow B(L^2(G))$ satisfying $\varphi(x \otimes y) - \lambda(x)\rho(y) \in K(L^2(G))$ (see [BO08, Proposition 15.1.4]) does not hold in this setting. Also the method used in [BO08, Section 15.2] can not be applied, since for a locally compact group G the crossed product $C(X) \rtimes_r G$ can be nuclear while $G \curvearrowright X$ is not amenable.

Corollary 4.6. *Let $G = B \rtimes_{\alpha} H$ be a semi-direct product of lcsc groups with B amenable. Then G is in class \mathcal{S} if and only if H is in class \mathcal{S} and there is a Borel map $\mu : B \rightarrow \text{Prob}(H)$ such that*

$$\lim_{b \rightarrow \infty} \|\mu(\alpha_h(b)) - h \cdot \mu(b)\| = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} \|\mu(aba') - \mu(b)\| = 0$$

uniformly on compact sets for $h \in H$ and $a, a' \in B$.

Proof. The only if part follows immediately from Proposition 4.4 and Remark 4.5. Conversely, let $\mu : B \rightarrow \text{Prob}(H)$ be a map as above. By Lemma 3.4, we can assume that μ is $\|\cdot\|$ -continuous. Let $H \curvearrowright^\beta B \times B$ be the diagonal action and let $G_0 = (B \times B) \rtimes_\beta H$. We apply Lemma 4.3 to the map μ with the morphism $\pi : G_0 \rightarrow G$ given by $\pi(a, a', h) = (a, h)$ for $a, a' \in B$ and $h \in H$, the space $X = B$, the action $G_0 \curvearrowright X$ given by $(a, a', h) \cdot b = a\alpha_h(b)(a')^{-1}$ for $a, a', b \in B$ and $h \in H$, and the amenable subgroup $K = B \subseteq G$. This yields a map $\tilde{\mu} : B \rightarrow \text{Prob}(G)$ as in Remark 4.5. Similarly, let $\eta : H \rightarrow \text{Prob}(H)$ be as in the definition of class \mathcal{S} . Applying Lemma 4.3 to the map ν with the morphism $\pi : G \times G \rightarrow G$ given by $\pi(g, g') = g$ for $g, g' \in G$, the space $X = H$, the action $G \times G \curvearrowright X$ induced by left and right translation and the amenable subgroup $K = B$, yields a map $\tilde{\nu} : H \rightarrow \text{Prob}(G)$ satisfying

$$\lim_{h \rightarrow \infty} \|\tilde{\nu}(khk') - (a, k) \cdot \tilde{\nu}(h)\| = 0$$

uniformly on compact sets for $k, k' \in G$ and $a \in B$. Now, the map $\nu : G \rightarrow \text{Prob}(G)$ defined by $\nu(b, h) = \tilde{\nu}(h)$ satisfies (4.3). \square

We are now ready to prove Theorem D. The suitable notion of wreath products for locally compact groups was introduced by Cornuier in [Cor17]. Let B and H be lcsc groups, X a countable set with continuous action $H \curvearrowright X$ and $A \subseteq B$ a compact open subgroup. The *semi-restricted power* $B^{X,A}$ is defined by

$$B^{X,A} = \{(b_x)_{x \in X} \in B^X \mid b_x \in A \text{ for all but finitely many } x \in X\}.$$

It is a lcsc space when equipped with the topology generated by the open sets $\prod_{x \in X} C_x$ where $C_x \subseteq B$ is open for every $x \in X$ and $C_x = A$ for all but finitely many $x \in X$. For $b \in B^{X,A}$, we denote $\text{supp}_A b = \{x \in X \mid b(x) \notin A\}$.

Denote by α the action of H on $B^{X,A}$ by translation, i.e. $\alpha_h(b)(x) = b(h^{-1}x)$ for $b \in B^{X,A}$, $h \in H$ and $x \in X$. It is easy to see that this action is continuous. The *(semi-restricted) wreath product* $B \wr_X^A$ is now defined as

$$B \wr_X^A H = B^{X,A} \rtimes_\alpha H \quad (4.7)$$

equipped with the product topology. By [Cor17, Proposition 2.4] it is a lcsc group. Theorem D is now a consequence of the following theorem.

Theorem 4.7. *Let A, B, X and H be as above. Suppose that B is non-compact and $|X| \geq 2$. Then, $B \wr_X^A H$ belongs to class \mathcal{S} if and only if B is amenable, the stabilizer $\text{Stab}_H(x)$ of every point $x \in X$ is amenable and H belongs to class \mathcal{S} .*

Note that if $|X| = 1$, then $B \wr_X^A H \cong B \times H$ belongs to class \mathcal{S} if and only if both factors are amenable, or one of B and H belongs to class \mathcal{S} and the other is compact. If B is compact, then by [BDV18, Lemma 7.2], we have that $B \wr_X^A H$ belongs to class \mathcal{S} if and only if H does.

Proof of Theorem 4.7. Suppose first that $B \wr_X^A H$ belongs to class \mathcal{S} . It follows that the subgroups H and $B \times B$ do. Hence, B must be amenable. For every point $x_0 \in X$ the subgroup

$$B \times \text{Stab}_H(x_0) \cong \{(b, h) \in B \wr_X^A G \mid b(x) = e \text{ if } x \neq x_0\}$$

belongs to class \mathcal{S} . Since B is non-compact, it follows that $\text{Stab}_H(x_0)$ is amenable.

Conversely, suppose that H belongs to class \mathcal{S} and that B and all stabilizers $\text{Stab}_H(x)$ are amenable. We prove that $B \wr_X^A H$ belongs to class \mathcal{S} . Denote by $X = \bigcup_{i \in I} X_i$ the partition of X into the orbits of $H \curvearrowright X$ and fix $x_i \in X_i$ for all $i \in I$. Write $B_i = B^{X_i, A}$ and $H_i = \text{Stab}_H(x_i)$.

Step 1. Each $B \wr_{X_i}^A H$ belongs to class \mathcal{S} . Fix $i \in I$. To prove this step, we proceed along the lines of [BO08, Corollary 15.3.6]. We claim that it suffices to prove the existence of a continuous map $\zeta_i : B_i \rightarrow M(H/H_i)^+ \cong \ell_1(X_i)^+$ satisfying

$$\lim_{b \rightarrow \infty} \frac{\|h \cdot \zeta_i(b) - \zeta_i(\alpha_h(b))\|_1}{\|\zeta_i(b)\|_1} = 0 \quad \text{and} \quad \lim_{b \rightarrow \infty} \frac{\|\zeta_i(aba') - \zeta_i(b)\|_1}{\|\zeta_i(b)\|_1} = 0 \quad (4.8)$$

uniformly on compact sets for $h \in H$ and $a, a' \in B_i$. Indeed, suppose that ζ_i is such a map. Let $H \curvearrowright^\beta B_i \times B_i$ be the diagonal action. We first normalize ζ_i and then apply Lemma 4.3 to this normalized map with the groups $G = (B_i \times B_i) \rtimes H$, H and $K = H_i$, the space $X = B_i$, the morphism $\pi : G \rightarrow H$ given by $\pi(a, a', h) = h$, and the

action $G \curvearrowright X$ given by $(a, a', h) \cdot b = a\alpha_h(b)(a')^{-1}$ for $a, a', b \in B_i$ and $h \in H$. This yields a map $\tilde{\zeta}_i : B_i \rightarrow \text{Prob}(H)$ that satisfies the conditions of Corollary 4.6.

By [Str74] every lsc group G admits a continuous proper length function, i.e. a continuous function $\ell : G \rightarrow \mathbb{R}^+$ such that $\ell(gh) \leq \ell(g) + \ell(h)$ and $\ell(g) = \ell(g^{-1})$ for all $g, h \in G$ and such that all the sets $\{g \in G \mid \ell(g) \leq M\}$ for $M > 0$ are compact. Fix such continuous, proper length functions $\ell_B : B \rightarrow \mathbb{R}^+$ and $\ell_H : H \rightarrow \mathbb{R}^+$. Define the function

$$f : X_i \rightarrow \mathbb{R}^+ : x \mapsto \inf_{\substack{h \in H \\ hx_i = x}} \ell_H(h).$$

Note that for every $M > 0$ the set $\{x \in X_i \mid f(x) \leq M\}$ is finite and that $f(hx) \leq \ell_H(h) + f(x)$. Similarly, we define

$$g : B \rightarrow \mathbb{R}^+ : b \mapsto \inf_{a, a' \in A} \ell_B(aba'),$$

and note that for every $M > 0$ the set $\{b \in B \mid g(b) \leq M\}$ is compact and that $g(bb') \leq g(b) + g(b') + N$, where $N = \sup_{a \in A} \ell_B(a)$. Also note that, by compactness of A , the map g is continuous.

Define $\zeta_i : B_i \rightarrow \ell^1(X_i)^+$ by

$$\zeta_i(b)(x) = \begin{cases} g(b(x)) + f(x) & \text{if } x \in \text{supp}_A(b), \\ 0 & \text{otherwise} \end{cases}$$

for $b \in B_i$ and $x \in X_i$.

We prove that ζ_i satisfies (4.8). Fix $h \in H$ and $a, a', b \in B_i$. Denote $b' = aba'$, $S = \text{supp}_A b$, $S' = \text{supp}_A b'$ and $T = \text{supp}_A a \cup \text{supp}_A a'$. We have

$$\|h \cdot \zeta(b) - \zeta(\alpha_h(b))\|_1 = \sum_{x \in hS} |f(h^{-1}x) - f(x)| \leq |S| \ell_H(h)$$

and

$$\begin{aligned} \|\zeta(b') - \zeta(b)\|_1 &= \sum_{x \in T} |\zeta(b')(x) - \zeta(b)(x)| \\ &= \sum_{x \in T \cap S \cap S'} |g(b'(x)) - g(b(x))| + \sum_{x \in (T \cap S) \setminus S'} |g(b(x)) + f(x)| + \sum_{x \in (T \cap S') \setminus S} |g(b'(x)) + f(x)| \\ &\leq \sum_{x \in T} |g(b'(x)) - g(b(x))| + \sum_{x \in T \cap (S \Delta S')} f(x) \\ &\leq \sum_{x \in T} \left(g(a'(x)) + g(a(x)) + 2N \right) + \sum_{x \in T \cap (S \Delta S')} f(x) \\ &\leq \|\zeta(a)\|_1 + \|\zeta(a')\|_1 + 2N |T| \end{aligned}$$

where we used in the third step that $g(b) = 0$ whenever $b \in A$.

So, it suffices to prove that

$$\lim_{b \rightarrow \infty} \|\zeta_i(b)\|_1 = +\infty \quad \text{and} \quad \lim_{b \rightarrow \infty} \frac{|\text{supp}_A b|}{\|\zeta_i(b)\|_1} = 0.$$

To prove the first, suppose that $\|\zeta(b)\|_1 \leq M$ for some $M > 0$. Then, $f(x) \leq M$ and $g(b(x)) \leq M$ for every $x \in \text{supp}_A(b)$. Hence,

$$b \in C = \prod_{x \in X_i} C_x$$

where $C_x = \{b \in B \mid g(b) \leq M\}$ for $x \in F = \{x \in X \mid f(x) \leq M\}$ and $C_x = A$ otherwise. Since F is finite and each C_x is compact, it follows that C is compact, which in turn implies the claim.

To prove that $|\text{supp}_A b|/\|\zeta_i(b)\|_1 \rightarrow 0$ if $b \rightarrow \infty$. Suppose that $|\text{supp}_A b|/\|\zeta_i(b)\|_1 \geq \delta$ for some $b \in B$ and $\delta > 0$. Denote $D = \{x \in X_i \mid f(x) \leq 2/\delta\}$. Then,

$$\frac{2}{\delta} (|\text{supp}_A b| - |D|) \leq \frac{2}{\delta} |\text{supp}_A b \setminus D| \leq \|\zeta_i(b)\|_1 \leq \frac{1}{\delta} |\text{supp}_A b|$$

and thus $|\text{supp}_A b| \leq 2|D|$. It follows that $\|\zeta_i(b)\|_1 \leq \frac{2}{\delta}|D|$. But, by the previous, the set

$$\left\{ b \in B \mid \|\zeta(b)\|_1 \leq \frac{2}{\delta}|D| \right\}$$

is compact and hence so is $\{b \in B \mid |\text{supp}_A b|/\|\zeta_i(b)\|_1 \geq \delta\}$.

Step 2. Construction of maps $\xi_i : B_i \rightarrow \text{Prob}(H)$ satisfying (4.9) below. Fix $i \in I$, $\varepsilon > 0$ and a compact $K \subseteq H$. In this step, we construct a Borel map $\xi_i : B_i \rightarrow \text{Prob}(H)$ such that

$$\|\xi_i(\alpha_h(b)) - h \cdot \xi_i(b)\| \leq \varepsilon \quad \text{and} \quad \xi_i(aba') = \xi_i(b) \quad (4.9)$$

for all $b \in B_i \setminus A^{X_i}$, all $h \in K$ and all $a, a' \in A^{X_i}$. Note that the difference with the previous step is that we want the map ξ_i to satisfy (4.9) for all $b \in B_i \setminus A^{X_i}$, instead of $b \in B_i \setminus L$ for L some (possibly large) compact set.

Since H_i is amenable, the action $H \curvearrowright H/H_i$ is amenable. Indeed, let $(\nu_n)_n$ be a sequence in $\text{Prob}(H_i)$ such that $\|h \cdot \nu_n - \nu_n\| \rightarrow 0$ uniformly on compact sets for $h \in H_i$ when $n \rightarrow \infty$. Fix a cross section $\sigma : H/H_i \rightarrow H$ for the quotient map $p : H \rightarrow H/H_i$. Then, the sequence of maps $\eta_n : H/H_i \rightarrow \text{Prob}(H)$ defined by

$$\eta_n(hH) = \sigma(hH) \cdot \nu_n$$

satisfies

$$\lim_{n \rightarrow \infty} \|h \cdot \eta_n(h'H) - \eta_n(hh'H)\| = \lim_{n \rightarrow \infty} \|\sigma(hh')^{-1} h \sigma(h'H) \cdot \nu_n - \nu_n\| = 0$$

uniformly on compact sets for $h \in H$ and $h'H \in H/H_i$.

By Proposition 2.5, it follows that we can take a sequence of maps such that the convergence holds uniformly on the whole of H/H_i . Hence, identifying $X_i \cong H/H_i$, we find a map $\mu : X_i \rightarrow \text{Prob}(H)$ such that

$$\|h \cdot \mu(x) - \mu(hx)\| < \varepsilon$$

for every $h \in K$ and every $x \in X_i$. Now, define $\xi_i : B_i \rightarrow \text{Prob}(H)$ by

$$\xi_i(b) = \frac{1}{|\text{supp}_A b|} \sum_{x \in \text{supp}_A b} \mu(x)$$

for $b \in B_i \setminus A^{X_i}$. For $b \in B_i$, set $\xi_i(b) = \mu_0$, where $\mu_0 \in \text{Prob}(H)$ is arbitrary. One easily checks that ξ_i satisfies (4.9).

Step 3. $B \wr_X^A H$ is bi-exact. Take $\varepsilon > 0$, a compact $C \subseteq B^{X,A}$ and a compact $K \subseteq H$. By Lemma 3.2 and Corollary 4.6, it suffices to prove that there exists a compact $D \subseteq B^{X,A}$ and a Borel map $\zeta : B^{X,A} \rightarrow \text{Prob}(H)$ such that

$$\|h \cdot \zeta(b) - \zeta(\alpha_h(b))\| \leq \varepsilon \quad \text{and} \quad \|\zeta(aba') - \zeta(b)\| \leq \varepsilon \quad (4.10)$$

for all $h \in K$, $a, a' \in C$ and $b \in B^{X,A} \setminus D$.

By definition of the topology on the semi-restricted product $B^{X,A}$, we can take compact sets $C_i \subseteq B_i$ for $i \in I$ such that

$$C \subseteq \prod_{i \in I} C_i$$

and such that $C_i = A^{X_i}$ for all but finitely many $i \in I$. Take $i_1, \dots, i_n \in I$ such that $C_i = A^{X_i}$ whenever $i \neq i_1, \dots, i_n$.

For $i = i_1, \dots, i_n$, the fact that $B \wr_X^A H$ belongs to class \mathcal{S} , allows us to take a compact $D_i \subseteq B_i$ and a Borel map $\zeta_i : B_i \rightarrow \text{Prob}(H)$ such that

$$\|h \cdot \zeta_i(b) - \zeta_i(\alpha_h(b))\| \leq \varepsilon \quad \text{and} \quad \|\zeta_i(aba') - \zeta_i(b)\| \leq \varepsilon$$

for $h \in K$, $a, a' \in C_i$ and $b \in B_i \setminus D_i$. By enlarging D_i , we can assume that $A^{X_i} \subseteq D_i$ and $C_i^{-1} A^{X_i} C_i^{-1} \subseteq D_i$. For $i \neq i_1, \dots, i_n$, we take for $\zeta_i : B_i \rightarrow \text{Prob}(H)$ the map ξ_i from step 2 and set $D_i = A^{X_i}$.

For $b \in B^{X,A}$ and $i \in I$, we denote by $b_i \in B^{X_i,A}$ the restriction of b to X_i . We also denote $I_b = \{i \in I \mid b_i \notin A^{X_i}\}$. Define $\zeta : B^{X,A} \rightarrow \text{Prob}(H)$ by

$$\zeta_i(b) = \frac{1}{|I_b|} \sum_{i \in I_b} \zeta_i(b_i)$$

for $b \in B^{X,A} \setminus A^X$ and $\zeta_i(b) = \delta_e$ for $b \in A^X$. One easily checks that (4.10) holds for $D = \prod_{i \in I} D_i$, since $I_b = I_{aba'}$ for $b \in B^{X,A} \setminus D$ and $a, a' \in C$. \square

For completeness, we also include a proof of the following fact mentioned in the introduction. It is a locally compact version of a result mentioned in [Oza06] in the countable setting.

Proposition 4.8. *A lsc group G that is inner amenable at infinity belongs to class \mathcal{S} if and only if G is amenable.*

Proof. If G is amenable, the result is immediate. Conversely, suppose that G is in class \mathcal{S} . Let $\eta : G \rightarrow \text{Prob}(G)$ be a map as in the definition. Define the map $\eta_* : C_b(G) \rightarrow C_b(G)$ by

$$(\eta_* f)(g) = \int_G f d\eta(g).$$

It is easy to prove that $\eta_*(\lambda_g f) - \lambda_g \eta_*(f) \in C_0(G)$ and $\eta_*(f) - \rho_g \eta_*(f) \in C_0(G)$ for all $f \in C_b(G)$ and $g \in G$.

Since G is inner amenable at infinity, we can take a state $m : C_b(G) \rightarrow \mathbb{C}$ that is invariant under conjugation and such that $m(f) = 0$ for all $f \in C_0(G)$. Then,

$$m \circ \eta_*(\lambda_g f) = m(\lambda_g \eta_*(f)) = m(\rho_g \lambda_g \eta_*(f)) = m \circ \eta_*(f).$$

Hence, $m \circ \eta_*$ is a left-invariant mean on $C_b(G)$. \square

5 Class \mathcal{S} is closed under measure equivalence

In this section, we prove Theorem E. As mentioned in the introduction, exactness is preserved under measure equivalence. So, it suffices to prove that property (S) (i.e. the existence of a map $\eta : G \rightarrow \text{Prob}(G)$ satisfying (1.1)) is a measure equivalence invariant. In order to prove that, we will use the characterization of measure equivalence in terms of cross section equivalence relations [KKR18, Theorem A] and introduce a notion of property (S) for these relations.

Recall that a countable, Borel equivalence relation \mathcal{R} on a standard probability space (X, μ) is an equivalence relation on X such that $\mathcal{R} \subseteq X \times X$ is a Borel subset and such that all orbits are countable. We say that \mathcal{R} is non-singular for the measure μ if $\mu(E) = 0$ implies that $\mu([E]_{\mathcal{R}}) = 0$ for all measurable $E \subseteq X$. Here, $[E]_{\mathcal{R}} = \{x \in X \mid \exists y \in E : x \sim_{\mathcal{R}} y\}$. We say that \mathcal{R} is *ergodic* if $E = [E]_{\mathcal{R}}$ implies that $\mu(E) = 0$ or $\mu(E) = 1$. We denote $\mathcal{R}^{(2)} = \{(x, y, z) \mid x \sim_{\mathcal{R}} y \sim_{\mathcal{R}} z\}$. Note that $\mathcal{R}^{(2)} \subseteq X \times X \times X$ is Borel.

A Borel subset $\mathcal{W} \subseteq \mathcal{R}$ is called *bounded* if the number of elements in its sections is bounded, i.e. if there exists a $C > 0$ such that

$$|{}_x \mathcal{W}| = |\{y \in X \mid (x, y) \in \mathcal{W}\}| < C \quad \text{and} \quad |\mathcal{W}_y| = |\{x \in X \mid (x, y) \in \mathcal{W}\}| < C$$

for a.e. $x, y \in X$. We say that \mathcal{W} is *locally bounded* if for every $\varepsilon > 0$, there exists a Borel subset $E \subseteq X$ with $\mu(X \setminus E) \leq \varepsilon$ such that $\mathcal{W} \cap (E \times E)$ is bounded.

The *full group* $[\mathcal{R}]$ is the group of all Borel automorphisms $\varphi : X \rightarrow X$, identified up to almost everywhere equality, such that $\text{graph } \varphi = \{(\varphi(x), x)\}_{x \in X}$ is contained in \mathcal{R} . The *full pseudo group* $[[\mathcal{R}]]$ is the set of all partial Borel isomorphisms $\varphi : A \rightarrow B$ for Borel sets $A, B \subseteq X$ whose graph is contained in \mathcal{R} . Again, these partial isomorphisms are identified up to almost everywhere equality. Every bounded Borel subset $\mathcal{W} \subseteq \mathcal{R}$ can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$. For more information about countable equivalence relations, see for instance [FM77].

Let G be a lsc group and $G \curvearrowright (X, \mu)$ a probability measure preserving (pmp) action. We say that the action $G \curvearrowright (X, \mu)$ is *essentially free* if the set

$$\{x \in X \mid \exists g \in G : gx = x\}$$

is a null set. Note that this set is Borel by [MRV13, Lemma 10].

The notion of a cross section equivalence relation was originally introduced by Forrest in [For74]. A more recent, self-contained treatment for unimodular groups can be found in [KPV15]. Given an essentially free pmp action $G \curvearrowright (X, \mu)$ on a standard probability space, a *cross section* is a Borel subset $X_1 \subseteq X$ with the following two properties.

- (i) There exists a neighborhood $U \subseteq G$ of identity such that the action map $U \times X_1 \rightarrow X : (g, x) \mapsto gx$ is injective.
- (ii) The subset $G \cdot X_1 \subseteq X$ is conull.

By [For74, Theorem 4.2] such a cross section always exists. Note that the first condition implies that the action map $\theta : G \times X_1 \rightarrow X : (g, x) \mapsto gx$ is countable-to-one and hence maps Borel sets to Borel sets. In particular, the set $G \cdot X_1$ in the second condition is Borel.

By removing a G -invariant null set from X , we can always assume that $G \cdot X_1 = X$ and that $G \curvearrowright X$ is really free. Hence, by [Kec95, 18.10 and 18.14], we can take a Borel map that is a right inverse of the map $G \times X_1 \rightarrow X : (g, x) \mapsto gx$. This yields Borel maps $\pi : X \rightarrow X_1$ and $\gamma : X \rightarrow G$ such that $x = \gamma(x) \cdot \pi(x)$ for all $x \in X$. Similarly, the map $G \times X \rightarrow X \times X : (g, x) \mapsto (gx, x)$ is injective and hence has a Borel image, which we denote by \mathcal{R}_G , and an inverse that is Borel. This yields a Borel map $\omega : \mathcal{R}_G \rightarrow G$ satisfying $\omega(x, y)y = x$ for $y \in G \cdot x$. Moreover, ω is a 1-cocycle in the sense that $\omega(x, y)\omega(y, z) = \omega(x, z)$ for all $y, z \in G \cdot x$.

The *cross section equivalence relation* associated to X_1 is defined by

$$\mathcal{R} = \mathcal{R}_G \cap (X_1 \times X_1) = \{(x, y) \in X_1 \times X_1 \mid y \in G \cdot X_1\}.$$

The measurable space X_1 admits a unique probability measure μ_1 and a unique number $0 < \text{covol}(X_1) < +\infty$ such that

$$(\lambda_G \otimes \mu_1)(\mathcal{W}) = \text{covol}(X_1) \int_X |\mathcal{W} \cap \theta^{-1}(x)| d\mu(x) \quad (5.1)$$

for all measurable $\mathcal{W} \subseteq G \times X_1$. The relation \mathcal{R} is a non-singular, countable, Borel equivalence relation for this probability measure μ_1 .

We will use the following easy lemma throughout the rest of this section.

Lemma 5.1. *Let G be a lsc group and $G \curvearrowright (X, \mu)$ an essentially free, pmp action. Let $X_1 \subseteq X$ be a cross section and \mathcal{R} the associated cross section equivalence relation. Then,*

- (a) *If $K \subseteq G$ is compact, then the set $\mathcal{W} = \{(x, y) \in \mathcal{R} \mid \omega(x, y) \in K\}$ is a bounded subset of \mathcal{R} .*
- (b) *If $\mathcal{W} \subseteq \mathcal{R}$ is a locally bounded set and $\varepsilon > 0$, then there exists a Borel subset $E \subseteq X_1$ with $\nu(E) < \varepsilon$ such that $\omega(\mathcal{W} \cap (E \times E))$ is relatively compact.*

Proof. Statement (a) follows easily from the fact that there is a neighborhood of the unit $e \in G$ for which the map $U \times X_1 \rightarrow X : (g, x) \mapsto gx$ is injective.

Since every bounded Borel subset can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$, it suffices to prove (b) for $\text{graph}(\varphi)$ with $\varphi \in [[\mathcal{R}]]$, but this can be done easily by taking $E = \{x \in X \mid \omega(\alpha(x), x) \in K\}$ for K a compact set that is large enough. \square

We define property (S) on the level of non-singular, countable, Borel equivalence relations as follows.

Definition 5.2. Let \mathcal{R} be a non-singular, countable, Borel equivalence relation on a standard measure space (X, μ) . We say that \mathcal{R} has *property (S)* if there exists a Borel map $\eta : \mathcal{R}^{(2)} \rightarrow \mathbb{C}$ such that

$$\sum_{\substack{z \in X \\ z \sim x}} \eta(x, y, z) = 1$$

for a.e. $(x, y) \in \mathcal{R}$ and such that for all $\varepsilon > 0$ and $\varphi, \psi \in [\mathcal{R}]$ the set

$$\left\{ (x, y) \in \mathcal{R} \mid \sum_{\substack{z \in X \\ z \sim x}} |\eta(x, y, z) - \eta(\varphi(x), \psi(y), z)| \geq \varepsilon \right\} \quad (5.2)$$

is locally bounded.

Remark 5.3. The map η above can be viewed as a map assigning to all $(x, y) \in \mathcal{R}$ a probability measure on the orbit of y such that for all $\varepsilon > 0$ and all $\varphi, \psi \in [\mathcal{R}]$ the set

$$\{(x, y) \in \mathcal{R} \mid \|\eta(\varphi(x), \psi(y)) - \eta(x, y)\|_1 \geq \varepsilon\} \quad (5.3)$$

is locally bounded.

We prove first that this notion of property (S) is stable under restrictions and amplifications of ergodic, countable equivalence relation.

Lemma 5.4. *Let \mathcal{R} be a countable, ergodic, non-singular equivalence relation on some standard probability space (X, μ) and let $X_0 \subseteq X$ be a Borel subset with positive measure. Then, \mathcal{R} has property (S) if and only if the restriction $\mathcal{R}_0 = \mathcal{R} \cap (X_0 \times X_0)$ has property (S).*

Proof. Since \mathcal{R} is ergodic, we can take a partition $Y = \bigcup_i Y_i$ and Borel isometries $\varphi_i \in [\mathcal{R}]$ such that $\varphi_i(Y_i) \subseteq Y_0$.

Suppose first that \mathcal{R}_0 has property (S) and let η_0 be as in Definition 5.2. We extend η_0 to a map η on \mathcal{R} by setting

$$\eta(x, y) = \eta_0(\varphi_i(x), \varphi_j(y))$$

for every $(x, y) \in \mathcal{R}$ with $x \in Y_i$ and $y \in Y_j$. It is straightforward to check that η satisfies (5.2) for every $\varepsilon > 0$ and $\varphi, \psi \in [\mathcal{R}]$.

Conversely, suppose that \mathcal{R} has property (S). Let η be a map as in the definition. Define for $(x, y) \in \mathcal{R}_0$ a probability measure on the \mathcal{R}_0 -orbit of y by setting

$$\eta_0(x, y)(z) = \sum_{\substack{i \in I \\ z \in \varphi_i(Y_i)}} \eta(x, y)(\varphi_i^{-1}(z))$$

whenever $(x, y) \in \mathcal{R}_0$. Clearly, η_0 satisfies (5.2) for every $\varepsilon > 0$ and every $\varphi, \psi \in [\mathcal{R}_0]$ □

Now, we prove that the above notion of property (S) is compatible with taking cross section equivalence relations.

Proposition 5.5. *Let G be a lsc group and $G \curvearrowright (X, \mu)$ an essentially free, ergodic, pmp action. Let $X_1 \subseteq X$ be a cross section and \mathcal{R} the associated cross section equivalence relation. Then, G has property (S) if and only if \mathcal{R} has property (S).*

Proof. As before, we fix Borel maps $\gamma : X \rightarrow G$ and $\pi : X \rightarrow X_1$ such that $x = \gamma(x) \cdot \pi(x)$ for a.e. $x \in X$. First, assume that G has property (S). Let $\eta : G \rightarrow \text{Prob}(G)$ be a map satisfying (1.1). Define for each $x \in X$ a map

$$\pi_x : G \rightarrow X_1 : g \mapsto \pi(g^{-1}x).$$

Note that π_x is a Borel map from G to the \mathcal{R} -orbit of $\pi(x)$. We define the map η' as in Definition 5.2 by

$$\eta'(x, y) = (\pi_x)_* \eta(\omega(x, y))$$

for $(x, y) \in \mathcal{R}$. Note that indeed every $\eta'(x, y)$ is a probability measure on the \mathcal{R} -orbit of x .

To prove that η' satisfies (5.3), fix $\varepsilon, \delta > 0$ and $\varphi, \psi \in [\mathcal{R}]$. It suffices to find a Borel set $E \subset X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that the set

$$\{(x, y) \in \mathcal{R} \cap (E \times E) \mid \|\eta'(\varphi(x), \psi(y)) - \eta'(x, y)\|_1 \geq \varepsilon\} \quad (5.4)$$

is bounded.

By Lemma 5.1, we find a compact set $K \subseteq G$ and a measurable $E \subseteq X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that $\omega(\varphi(x), x) \in K$ and $\omega(y, \psi(y)) \in K$ for all $x, y \in E$. Take a compact set $L \subset G$ such that $\|\eta(gkh) - g \cdot \eta(k)\|_1 < \varepsilon$ for all $g, h \in K$ and all $k \in G \setminus L$. We claim that

$$\|\eta'(\varphi(x), \psi(y)) - \eta'(x, y)\|_1 < \varepsilon \quad (5.5)$$

whenever $(x, y) \in \mathcal{R} \cap (E \times E)$ and $\omega(x, y) \in G \setminus L$. Assuming the claim is true, the set (5.4) is contained in the set of all $(x, y) \in \mathcal{R}$ with $\omega(x, y) \in L$ which is bounded by Lemma 5.1. To prove (5.5), fix $(x, y) \in \mathcal{R} \cap (E \times E)$ with $\omega(x, y) \in G \setminus L$. We have

$$\|\eta'(\varphi(x), \psi(y)) - \eta'(x, y)\|_1 = \|(\pi_{\varphi(x)})_* \eta(\omega(\varphi(x), \psi(y))) - (\pi_x)_* \eta(\omega(x, y))\|_1$$

Now, $\pi_x(g) = \pi_{\varphi(x)}(\omega(\varphi(x), x)g)$ and hence

$$(\pi_x)_*(\eta(\omega(x, y))) = (\pi_{\varphi(x)})_*(\omega(\varphi(x), x) \cdot \eta(\omega(x, y)))$$

which yields that

$$\|\eta'(\varphi(x), \psi(y)) - \eta'(x, y)\|_1 \leq \|\eta(\omega(\varphi(x), \psi(y))) - \omega(\varphi(x), x) \cdot \eta(\omega(x, y))\|_1 < \varepsilon.$$

where we used the identity

$$\omega(\varphi(x), \psi(y)) = \omega(\varphi(x), x)\omega(x, y)\omega(y, \psi(y))$$

and the assumption that $\omega(\varphi(x), x), \omega(y, \psi(y)) \in K$ and $\omega(x, y) \in G \setminus L$. Hence, (5.5) is proved.

Conversely, assume that \mathcal{R} has property (S) and let η be a map as in the definition. Choose an arbitrary $\xi \in \text{Prob}(G)$ and define

$$\eta' : G \rightarrow \text{Prob}(G) : g \mapsto \int_X \left(\sum_{\substack{z \in X_1 \\ z \sim \pi(x)}} \eta(\pi(gx), \pi(x), z)\omega(gx, z) \cdot \xi \right) d\mu(x).$$

We prove that η' satisfies (1.1). To motivate the arbitrary choice of ξ , note that whenever η' satisfies (1.1), so does the map $g \mapsto \eta'(g) * \xi$, where $\eta'(g) * \xi$ denotes the convolution product of $\eta'(g), \xi \in \text{Prob}(G)$.

Fix a symmetric, compact neighborhood K of the unit e in G and an $\varepsilon > 0$. Take a compact, symmetric subset $L \subseteq G$ such that $F = \gamma^{-1}(L)$ satisfies $\mu(F) \geq 1 - \varepsilon$. Denote $\kappa = \lambda_G(L)/\text{covol}(X_1)$. By Lemma 5.1, the set

$$\mathcal{W} = \{(x, y) \in \mathcal{R} \mid \omega(x, y) \in LKL\}$$

is bounded Borel. Writing \mathcal{W} as a union of finitely many elements of $[[\mathcal{R}]]$ and using (5.3), we see that the set

$$\mathcal{V} = \{(x, y) \in \mathcal{R} \mid \exists (x, x'), (y, y') \in \mathcal{W}, \|\eta(x', y') - \eta(x, y)\|_1 \geq \varepsilon\}$$

is locally bounded. Denoting $\delta = \varepsilon/\kappa$ and using Lemma 5.1, we can find a compact set $C \subseteq G$ and a measurable $E \subseteq X_1$ and with $\mu_1(E) \geq 1 - \delta$ such that $\omega(\mathcal{V} \cap (E \times E)) \subseteq C$. We conclude that

$$\|\eta(x', y') - \eta(x, y)\|_1 < \varepsilon \tag{5.6}$$

whenever $(x, y) \in \mathcal{R} \cap (E \times E)$ with $(x, x') \in \mathcal{W}, (y, y') \in \mathcal{W}$ and $\omega(x, y) \in G \setminus C$.

Denote $D = LCL$. We conclude the proposition by proving that

$$\|\eta'(gkh) - g \cdot \eta'(k)\| < 4\kappa\delta + 9\varepsilon = 13\varepsilon \tag{5.7}$$

for all $g, h \in K$ and $k \in G \setminus D$. So, fix $g, h \in K$ and $k \in G \setminus D$. Applying the change of variables $x \mapsto h^{-1}x$ and using that $\omega(gkx, z) = g\omega(kx, z)$, we find that

$$\eta'(gkh) = g \cdot \left(\int_X \left(\sum_{\substack{z \in X_1 \\ z \sim \pi(x)}} \eta(\pi(gkx), \pi(h^{-1}x), z)\omega(kx, z) \cdot \xi \right) d\mu(x) \right)$$

and hence

$$\|\eta'(gkh) - g \cdot \eta'(k)\| \leq \int_X \|\eta(\pi(gkx), \pi(h^{-1}x)) - \eta(\pi(kx), \pi(x))\|_1 d\mu(x).$$

Since $g, h^{-1} \in K$, we have that $(\pi(gkx), \pi(kx)) \in \mathcal{W}$ and $(\pi(h^{-1}x), \pi(x)) \in \mathcal{W}$ whenever $x \in X$ is such that $gkx, h^{-1}x, kx, x \in F = \gamma^{-1}(L)$. Moreover, for such an x we also have $\omega(\pi(kx), \pi(x)) \in LkL \subseteq G \setminus C$. Hence, by (5.6) we have that

$$\|\eta(\pi(gkx), \pi(h^{-1}x)) - \eta(\pi(kx), \pi(x))\|_1 < \varepsilon \tag{5.8}$$

whenever $gkx, h^{-1}x, kx, x \in F, \pi(x) \in E$ and $\pi(kx) \in E$.

Since $\mu(F) \geq 1 - \varepsilon$, we can find a measurable set F' with $\mu(F') \geq 1 - 4\varepsilon$ such that $gkx, h^{-1}x, kx, x \in F$ for every $x \in F'$. Moreover, the map $\theta : G \times X_1 \rightarrow X$ is injective on the image A of the map $x \mapsto (\gamma(x), \pi(x))$. Hence

by (5.1), we have that $\text{covol}(X_1) \mu(\theta(\mathcal{U})) = (\lambda_G \otimes \mu_1)(\mathcal{U})$ for all $\mathcal{U} \subseteq A$. It follows that for measurable $S \subseteq X_1$, we have that

$$\mu(\pi^{-1}(S) \cap F) = \text{covol}(X_1)^{-1}(\lambda_G \times \mu_1)(A \cap (L \times S)) \leq \frac{\lambda_G(L)}{\text{covol}(X_1)} \mu_1(S) = \kappa \mu_1(S).$$

Applying this to $\pi^{-1}(X_1 \setminus E) \cap F$ and using the definition F' above, we conclude that (5.8) holds on a set whose complement has at most measure $4\varepsilon + 2\kappa\delta$ and hence that (5.7) holds. \square

The proof of Theorem E is now easy.

Proof of Theorem E. Let G be a lcsc group in class \mathcal{S} and let H be a lcsc group that is measure equivalent to G . As mentioned in the introduction, we have that H is exact. Indeed, by [DL15, Corollary 2.9] and [BCL17, Theorem A] G is exact if and only if the proper metric space (G, d) has property (A) in the sense of Roe, where d is any proper left-invariant metric that implements the topology on G , and by [DL14, Theorem 0.1 (6)], property A is a measure equivalence invariant.

By [KKR18, Theorem A] and [KKR17, Theorem A], G and H admit free, ergodic, probability measure preserving actions $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ with cross sections $X_1 \subseteq X$, $Y_1 \subseteq Y$ and cross section equivalence relations \mathcal{R} and \mathcal{T} respectively such that \mathcal{R} is stably isomorphic to \mathcal{T} . But, by Proposition 5.5, the relation \mathcal{R} (resp. \mathcal{T}) has property (S) if and only if G (resp. H) has and by Lemma 5.4, \mathcal{R} has property (S) if and only if \mathcal{T} has. \square

6 Class \mathcal{S} and unique prime factorization

In [HI17], Houdayer and Isono introduce the following property.

Definition 6.1. Let $(M, \mathcal{H}, J, \mathfrak{F})$ be a von Neumann algebra in standard form. We say that M satisfies the *strong condition (AO)* if there exist C^* -algebras $A \subseteq M$ and $\mathcal{C} \subseteq B(\mathcal{H})$ such that

- A is exact and σ -weakly dense in M ,
- \mathcal{C} is nuclear and contains A ,
- all commutators $[c, JaJ]$ for $c \in \mathcal{C}$ and $a \in A$ belong to the compact operators $K(\mathcal{H})$.

Note that the definition in [HI17, Definition 2.6] also requires A and \mathcal{C} to be unital. However, by [BO08, Proposition 2.2.1 and Proposition 2.2.4] this requirement is not essential.

In [HI17, Theorems B], Houdayer and Isono provide a unique prime factorization theorem for non-amenable factors satisfying strong condition (AO). A slightly more general version, removing the condition that the unknown tensor product factors N_i have a state with large centralizers, was later proved by Ando, Haagerup, Houdayer, and Marrakchi in [AHHM18, Application 4]. Theorem F now follow immediately by combining these theorems with the following result.

Proposition 6.2. *Let G be a lcsc group in class \mathcal{S} , then its group von Neumann algebra $L(G)$ satisfies strong condition (AO).*

Proof. Recall that $L(G)$ is in standard form on $L^2(G)$. The anti-unitary operator J is given by

$$(J\xi)(t) = \delta_G(t)^{-1/2} \overline{\xi(t^{-1})},$$

where δ_G denotes the modular function of G . The action of an element $\lambda(f) \in L(G)$ for $f \in C_c(G)$ is given by

$$(\lambda(f)\xi)(s) = \int_G f(t) \xi(t^{-1}s) dt.$$

Straightforward calculation yields

$$(J\lambda(f)J\xi)(s) = \int_G \overline{f(t)} \delta_G(t)^{1/2} \xi(st) dt.$$

Let $A = C_r^*(G)$ be the reduced group C^* -algebra of G . Then, obviously A is exact and σ -weakly dense in $L(G)$. By Theorem B and [Ana02, Theorem 5.3], the algebra $C(h^u G) \rtimes G$ is nuclear. Now, the inclusion $C(h^u G) \subseteq C_b^u(G) \hookrightarrow$

$B(L^2(G))$ together with the unitary representation $g \mapsto \lambda_g$ induces a *-morphism $\pi : C(h^u G) \rtimes G \rightarrow B(L^2(G))$. Let \mathcal{C} be the image of this *-morphism. The algebra \mathcal{C} is nuclear as a quotient of a nuclear C^* -algebra, and obviously contains A . Note that $C_c(G, C(h^u G))$ is a dense subalgebra in $C(h^u G) \rtimes G$. Identifying an element $h \in C_c(G, C(h^u G)) \subseteq C(h^u G) \rtimes G$ with a function on $G \times G$ that is compactly supported in the first component, we get that the action $\pi(h)$ on a $\xi \in L^2(G)$ is given by

$$(\pi(h)\xi)(s) = \int_G h(t, s)\xi(t^{-1}s) dt.$$

Denote by \mathcal{C}_0 the image of $C_c(G, C(h^u G))$ under π .

We prove that \mathcal{C} commutes with JAJ up to the compact operators. Since $C_c(G)$ is dense in $C_r^*(G)$ and \mathcal{C}_0 is dense in \mathcal{C} , it suffices to prove that for every $f \in C_c(G)$ and every $h \in C_c(G, C(h^u G))$, we have $T = [\pi(h), J\lambda(f)J] \in K(L^2(G))$. A straightforward calculation yields that for $\xi \in L^2(G)$ and $s \in G$, we have

$$(T\xi)(s) = \int_G \int_G (h(t, s) - h(t, su)) \overline{f(u)} \delta_G(u)^{1/2} \xi(t^{-1}su) dt du$$

Let $(K_n)_n$ be an increasing sequence of compact subsets of G such that $G = \bigcup_n K_n$. Take a compact $L \subseteq G$ that contains the support of f and of (the first component of) h . Define the operator $T_n \in B(L^2(G))$ by

$$\begin{aligned} (T_n\xi)(s) &= \int_G \int_G \chi_{K_n}(s) (h(t, s) - h(t, su)) \overline{f(u)} \delta_G(u)^{1/2} \xi(t^{-1}su) dt du \\ &= \int_G \int_G \chi_{K_n}(s) (h(t, s) - h(t, tu)) \overline{f(s^{-1}tu)} \delta_G(s^{-1}tu)^{1/2} \xi(u) dt du \\ &= \int_G k_n(s, u) \xi(u) du \end{aligned}$$

where

$$k_n(s, u) = \chi_{K_n}(s) \int_G (h(t, s) - h(t, tu)) \overline{f(s^{-1}tu)} \delta_G(s^{-1}tu)^{1/2} dt.$$

Note that since $f \in C_c(G)$ and h is compactly supported in the first component, we have that each $k_n \in L^2(G \times G)$ and hence that T_n is compact. Moreover, $T_n \rightarrow T$ in norm since

$$\begin{aligned} \|T\xi - T_n\xi\|^2 &= \int_{G \setminus K_n} \left| \int_G \int_G (h(t, s) - h(t, su)) \overline{f(u)} \delta_G(u)^{1/2} \xi(t^{-1}su) dt du \right|^2 ds \\ &\leq \int_{G \setminus K_n} \sup_{t, u \in L} |h(t, s) - h(t, su)|^2 \left(\int_L \int_L |f(u)| \delta_G(u)^{1/2} |\xi(t^{-1}su)| du dt \right)^2 ds \\ &\leq \sup_{s \in G \setminus K_n} \sup_{t, u \in L} |h(t, s) - h(t, su)|^2 \mu(L)^2 \|J\lambda(|f|)J|\xi\|_2^2 \\ &= \sup_{s \in G \setminus K_n} \sup_{t, u \in L} |h(t, s) - h(t, su)|^2 \mu(L)^2 \|f\|_1^2 \|\xi\|_2^2 \end{aligned}$$

and

$$\limsup_{s \rightarrow \infty} |h(t, s) - h(t, su)|^2 = 0$$

uniformly on compact sets for $t, u \in G$. We conclude that T itself is compact. \square

References

- [AD08] G. Arzhantseva and T. Delzant, Examples of random groups. 2008. *Preprint*. Available at: <http://www.unige.ch/math/folks/arjantse/Abs/random.pdf>.
- [Ada94] S. Adams, Boundary amenability for word hyperbolic groups and an application to smooth dynamics of simple groups. *Topology* **33** (1994), 765–783.

- [AHHM18] H. Ando, U. Haagerup, C. Houdayer, and A. Marrakchi, Structure of bicentralizer algebras and inclusions of type III factors. 2018. *Preprint*. arXiv: [1804.05706](https://arxiv.org/abs/1804.05706).
- [Ana02] C. Anantharaman-Delaroche, Amenability and exactness for dynamical systems and their C^* -algebras. *Trans. Am. Math. Soc.* **354** (2002), 4153–4179.
- [BCL17] J. Brodzki, C. Cave, and K. Li, Exactness of locally compact groups. *Adv. Math.* **312** (2017), 209–233.
- [BDV18] A. Brothier, T. Deprez, and S. Vaes, Rigidity for von Neumann algebras given by locally compact groups and their crossed products. *Commun. Math. Phys.* **361** (2018), 85–125.
- [BO08] N. Brown and N. Ozawa, C^* -Algebras and Finite-Dimensional Approximations. *Graduate Studies in Mathematics* **88**. American Mathematical Society, Providence, 2008.
- [CdSS16] I. Chifan, R. de Santiago, and T. Sinclair, W^* -rigidity for the von Neumann algebras of products of hyperbolic groups. *Geom. Funct. Anal.* **26** (2016), 136–159.
- [CI18] I. Chifan and A. Ioana, Amalgamated free product rigidity for group von Neumann algebras. *Adv. Math.* **329** (2018), 819–850.
- [Con76] A. Connes, Classification of injective factors. Cases II_1 , II_∞ , III_λ , $\lambda \neq 1$. *Ann. Math.* **104** (1976), 73–115.
- [Cor17] Y. Cornuier, Locally compact wreath products. 2017. *Preprint*. arXiv: [1703.08880](https://arxiv.org/abs/1703.08880). To appear in *J. Aust. Math. Soc.*
- [CS13] I. Chifan and T. Sinclair, On the structural theory of II_1 factors of negatively curved groups. *Ann. Sci. Ecole Norm. S.* **46** (2013), 1–33.
- [DL14] S. Deprez and K. Li, Permanence properties of property A and coarse embeddability for locally compact groups. 2014. *Preprint*. arXiv: [1403.7111](https://arxiv.org/abs/1403.7111).
- [DL15] S. Deprez and K. Li, Property A and uniform embedding for locally compact groups. *J. Noncommut. Geom.* **9** (2015), 797–819.
- [FM77] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I. *Trans. Am. Math. Soc.* **234** (1977), 289–324.
- [For74] P. Forrest, On the virtual groups defined by ergodic actions of \mathbb{R}^n and \mathbb{Z}^n . *Adv. Math.* **14** (1974), 271–308.
- [GHW05] E. Guentner, N. Higson, and S. Weinberger, The Novikov conjecture for linear groups. *Publ. Math. Inst. Hautes Études Sci.* **101** (2005), 243–268.
- [Gro03] M. Gromov, Random walk in random groups. *Geom. Funct. Anal.* **13** (2003), 73–146.
- [HI17] C. Houdayer and Y. Isono, Unique prime factorization and bicentralizer problem for a class of type III factors. *Adv. Math.* **305** (2017), 402–455.
- [HK94] U. Haagerup and J. Kraus, Approximation properties for group C^* -algebras and group von Neumann algebras. *Trans. Am. Math. Soc.* **344** (1994), 667–699.
- [HR19] C. Houdayer and S. Raum, Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras. *Comment. Math. Helv.* **94** (2019), 185–219.
- [HV13] C. Houdayer and S. Vaes, Type III factors with unique Cartan decomposition. *J. Math. Pures Appl.* **100** (2013), 564–590.
- [Kec95] A. S. Kechris, Classical Descriptive Set Theory. *Graduate Texts in Mathematics* **156**. Springer, New York, 1995.
- [KKR17] J. Koivisto, D. Kyed, and S. Raum, Measure equivalence and coarse equivalence for unimodular locally compact groups. 2017. *Preprint*. arXiv: [1703.08121](https://arxiv.org/abs/1703.08121).
- [KKR18] J. Koivisto, D. Kyed, and S. Raum, Measure equivalence for non-unimodular groups. 2018. *Preprint*. arXiv: [1805.09063](https://arxiv.org/abs/1805.09063).
- [KPV15] D. Kyed, H. D. Petersen, and S. Vaes, L^2 -Betti numbers of locally compact groups and their cross section equivalence relations. *Trans. Am. Math. Soc.* **367** (2015), 4917–4956.
- [KW99a] E. Kirchberg and S. Wassermann, Exact groups and continuous bundles of C^* -algebras. *Math. Ann.* **315** (1999), 169–203.

- [KW99b] E. Kirchberg and S. Wassermann, Permanence properties of C^* -exact groups. *Doc. Math.* **4** (1999), 513–558.
- [Mac52] G. W. Mackey, Induced representations of locally compact groups I. *Ann. Math.* **55** (1952), 101–139.
- [MRV13] N. Meesschaert, S. Raum, and S. Vaes, Stable orbit equivalence of Bernoulli actions of free groups and isomorphism of some of their factor actions. *Expo. Math.* **31** (2013), 274–294.
- [OP04] N. Ozawa and S. Popa, Some prime factorization results for type II_1 factors. *Invent. Math.* **156** (2004), 223–234.
- [Osa14] D. Osajda, Small cancellation labellings of some infinite graphs and applications. 2014. *Preprint*. arXiv: [1406.5015](https://arxiv.org/abs/1406.5015).
- [Oza04] N. Ozawa, Solid von Neumann algebras. *Acta Math.* **192** (2004), 111–117.
- [Oza06] N. Ozawa, A Kurosh-type theorem for type II_1 factors. *Int. Math. Res. Not.* (2006), 1–21.
- [Oza09] N. Ozawa, An example of a solid von Neumann algebra. *Hokkaido Math. J.* **38** (2009), 557–561.
- [PV14] S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups. *J. Reine Angew. Math.* **694** (2014), 141–198.
- [Rau19a] S. Raum, C^* -simplicity after Breuillard, Haagerup, Kalantar, Kennedy and Ozawa. 2019. URL: <http://raum-brothers.eu/sven/data/publications/bourbaki.pdf>.
- [Rau19b] S. Raum, C^* -simplicity of locally compact Powers groups. *J. Reine Angew. Math.* **748** (2019), 173–205.
- [Sak09] H. Sako, The class \mathcal{S} as an ME invariant. *Int. Math. Res. Not.* (2009), 2749–2759.
- [Ska88] G. Skandalis, Une notion de nucléarité en K -théorie (d’après J. Cuntz). *K-Theory* **1** (1988), 549–573.
- [Str74] R. A. Struble, Metrics in locally compact groups. *Compos. Math.* **28** (1974), 217–222.
- [Suz16] Y. Suzuki, Elementary constructions of non-discrete C^* -simple groups. *Proc. Am. Math. Soc.* **145** (2016), 1369–1371.