Ozawa's class $\mathcal{S}$ for locally compact groups and unique prime factorization of group von Neumann algebras

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Abstract

We study class $\mathcal{S}$ for locally compact groups. We characterize locally compact groups in this class as groups having an amenable action on a boundary that is small at infinity, generalizing a theorem of Ozawa. Using this characterization, we provide new examples of groups in class $\mathcal{S}$ and prove a unique prime factorization theorem for group von Neumann algebras of products of locally compact groups in this class. We also prove that class $\mathcal{S}$ is a measure equivalence invariant.

1 Introduction

Class $\mathcal{S}$ for countable groups was introduced by Ozawa in [Oza06]. A countable group Γ is said to be in *class* $\mathcal{S}$ if it is exact and it admits a map $\eta : \Gamma \to \mathrm{Prob}(\Gamma)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0$$

for all $g, h \in \Gamma$. Equivalently, class $\mathcal{S}$ can be characterized as the class of all groups that admit an amenable action on a boundary that is *small at infinity* (see [Oza06, Theorem 4.1]). Groups in class $\mathcal{S}$ are also called *bi-exact*.

Class $\mathcal{S}$ is used in, among others, [Oza04; Oza06; OP04; CS13; PV14; CI18; CdSS16] to prove rigidity results for group von Neumann algebras of countable groups. In [Oza04], Ozawa proved that the group von Neumann algebra $L(\Gamma)$ is solid when Γ belongs to class $\mathcal{S}$. This implies in particular that for Γ icc, non-amenable and in class $\mathcal{S}$, the group von Neumann algebra $L(\Gamma)$ is *prime*, i.e. $L(\Gamma)$ does not decompose as a tensor product $M_1 \otimes M_2$ for non-type I factors M_1 and M_2.

In [OP04], Ozawa and Popa proved the first unique prime factorization results for von Neumann algebras using groups in this class. Among other results, they showed that if $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ is a product of non-amenable, icc groups in class $\mathcal{S}$, then $L(\Gamma) \cong L(\Gamma_1) \otimes \cdots \otimes L(\Gamma_n)$ remembers the number of factors n and each factor $L(\Gamma_i)$ up to amplification, i.e. if $L(\Gamma) \cong N_1 \otimes \cdots \otimes N_m$ for some prime factors $N_1, \dots, N_m$, then $n = m$ and (after relabeling) $L(\Gamma_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$. Subclasses of class $\mathcal{S}$ were used in [CS13; PV14; HV13] to prove rigidity results on crossed product von Neumann algebras $L^\infty(X) \rtimes \Gamma$.

Examples of countable groups in class $\mathcal{S}$ are amenable groups, hyperbolic groups (see [Ada94]), lattices in connected simple Lie groups of real rank one (see [Ska88, Proof of Théorème 4.4]), wreath products $B \wr \Gamma$ with B amenable and Γ in class $\mathcal{S}$ (see [Oza06]) and $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ (see [Oza09]). Moreover, class $\mathcal{S}$ is closed under measure equivalence (see [Sak09]). Examples of groups not belonging to class $\mathcal{S}$ are product groups $\Gamma \times \Lambda$ with Γ non-amenable and Λ infinite, non-amenable inner amenable groups and non-amenable groups with infinite centre.

In this paper, we study class $\mathcal{S}$ for locally compact groups. We provide a characterization of groups in this class similar to [Oza06, Theorem 4.1], we provide new examples of groups in this class and we prove a unique prime factorization result for group von Neumann algebras of locally compact groups. We also prove that class $\mathcal{S}$ is a measure equivalence invariant.

Let G be a locally compact second countable (lcsc) group. We denote by $\mathrm{Prob}(G)$ the space of all Borel probability measures, i.e. the state space of $C_0(G)$. The precise definition of class $\mathcal{S}$ for locally compact groups is now as follows.

Definition A. Let G be a lcsc group. We say that G is in *class* $\mathcal{S}$ (or *bi-exact*) if G is exact and if there exists a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0 \tag{1.1}$$

uniformly on compact sets for $g, h \in G$.

In [BDV18] this property without the exactness condition was called *property (S)*. Note that the definition was slightly different: the image of the map η above was in the space $\mathcal{S}(G) = \{f \in L^1(G)^+ \mid \|f\|_1 = 1\}$ instead of $\mathrm{Prob}(G)$. However, we prove in Proposition 3.1 that this is equivalent. It is also worthwhile to note that it is currently unknown whether there are groups with property (S) that are not exact.

Examples of lcsc groups in class $\mathcal{S}$ include amenable groups, groups acting continuously and properly on a tree or hyperbolic graph of uniformly bounded degree, and connected, simple Lie groups of real rank one with finite centre. Proofs of these results can be found in [BDV18, Section 7]. It is easy to prove that groups not in class $\mathcal{S}$ include product groups $G \times H$ with G non-amenable and H non-compact, non-amenable groups G with non-compact centre and non-amenable groups G that are *inner amenable at infinity*, i.e. for which there exists a conjugation invariant mean m on G such that $m(E) = 0$ for every compact set $E \subseteq G$.

Given a locally compact group G, we denote by $C_b^u(G)$ the algebra of *bounded uniformly continuous* functions on G, i.e. the bounded functions $f : G \to \mathbb{C}$ such that

$$\|\lambda_g f - f\|_\infty \to 0 \qquad \text{and} \qquad \|\rho_g f - f\|_\infty \to 0$$

whenever $g \to e$. Here, λ and ρ denote the left and right regular representations defined by $(\lambda_g f)(h) = f(g^{-1}h)$ and $(\rho_g f)(h) = f(hg)$ respectively. We define the compactification h^uG of the group G as the spectrum of the following algebra

$$C(h^uG) \cong \{f \in C_b^u(G) \mid \rho_g f - f \in C_0(G) \text{ for all } g \in G\}$$

and denote by $\nu^uG = h^uG \setminus G$ its boundary. The compactification h^uG is equivariant in the sense that both actions $G \curvearrowright G$ by left and right translation extend to continuous actions $G \curvearrowright h^uG$. It is also *small at infinity* in the sense that the extension of the action by right translation is trivial on the boundary ν^uG. It is moreover the universal equivariant compactification that is small at infinity, in the sense that for every equivariant compactification $\overline{G}$ that is small at infinity, the inclusion $G \hookrightarrow \overline{G}$ extends to a continuous G-equivariant map $h^uG \to \overline{G}$.

The following locally compact version of [Oza06, Theorem 4.1], characterizes groups in class $\mathcal{S}$ as groups acting amenably on the boundary ν^uG.

Theorem B. *Let G be a lcsc group. Then, the following are equivalent*

(i) *G is in class $\mathcal{S}$,*

(ii) *the action $G \curvearrowright \nu^uG$ induced by left translation is topologically amenable,*

(iii) *the action $G \curvearrowright h^uG$ induced by left translation is topologically amenable,*

(iv) *the action $G \times G \curvearrowright C_b^u(G)/C_0(G)$ induced by left and right translation is topologically amenable.*

The two novelties in the proof of this result are the proof of (iii) and the method we used to prove the implication (iv)⇒(i). Indeed, the original proof of Ozawa for countable groups used that G belongs to class $\mathcal{S}$ if and only if there is a u.c.p map $\theta : \mathrm{C}_r^*(G) \otimes_{\min} \mathrm{C}_r^*(G) \to B(L^2(G))$ satisfying $\theta(x \otimes y) - \lambda(x)\rho(y) \in K(L^2(G))$, where λ and ρ denote the representations of $\mathrm{C}_r^*(G)$ induced by the left and right regular representation, respectively. This is however no longer true for locally compact groups. Indeed, for all connected groups G, the reduced C*-algebra $\mathrm{C}_r^*(G)$ is nuclear and hence a map θ as above always exists.

Denote by $\beta^{lu}G$ the left-equivariant Stone-Čech compactification of G, i.e. the spectrum of the algebra $C_b^{lu}(G)$ of bounded left-uniformly continuous functions on G. The action $G \curvearrowright G$ by left-translation extends uniquely to a continuous action $G \curvearrowright \beta^{lu}G$. Moreover, $\beta^{lu}G$ is the universal left-equivariant compactification of G in the sense that every left-G-equivariant continuous map $G \to X$ to any compact space X with continuous action $G \curvearrowright X$ extends uniquely to a G-equivariant continuous map $\beta^{lu}G \to X$. We also prove the following characterization of groups in class $\mathcal{S}$.

Theorem C. *Let G be a lcsc group. Then, G belongs to class $\mathcal{S}$ if and only if G is exact and there exists a Borel map $\eta : G \to \mathrm{Prob}(\beta^{lu}G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

In the proof of this theorem, we will see that it is precisely the exactness of G that allows us to construct the required map $\tilde{\eta} : G \to \mathrm{Prob}(G)$ from a map $\eta : G \to \mathrm{Prob}(\beta^{lu}G)$. This was implicitly observed before in [BO08, Chapter 15] for countable groups.

Using Theorem B, we prove the following new examples of locally compact groups in class $\mathcal{S}$. In [Cor17], Cornulier introduced a notion of wreath products for locally compact groups. See (4.5) on page 12 for a short recapitulation and the notation used in this article. The following result is a locally compact version of [Oza06, Corollary 4.5].

Theorem D. *Let B and H be lcsc groups, X a countable set with a continuous action $H \curvearrowright X$ and $A \subseteq B$ a compact open subgroup. If B is amenable, all stabilizers $\mathrm{Stab}_H(x)$ for $x \in X$ are amenable and H belongs to class $\mathcal{S}$, then also the wreath product $B \wr_X^A H$ belongs to class $\mathcal{S}$.*

A notion of measure equivalence for locally compact groups was introduced by S. Deprez and Li in [DL14]. By [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)] exactness is preserved under this notion of measure equivalence. More recently, this notion was studied in more detail in [KKR17; KKR18]. It was proved that two lcsc groups G and H are measure equivalent if and only if they admit essentially free, ergodic pmp actions on some standard probability space for which the cross section equivalence relations are stably isomorphic. Using this characterization, we were able to prove the following result. For countable groups this was proven by Sako in [Sak09].

Theorem E. *The class $\mathcal{S}$ is closed under measure equivalence.*

As a consequence of this theorem, we have for instance that $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ belong to class $\mathcal{S}$. Indeed, $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a lattice in both $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. Hence, the latter two are measure equivalent to $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$, which belongs to class $\mathcal{S}$ by [Oza09].

In [BDV18], the author proved together with Brothier and Vaes that the group von Neumann algebra $L(G)$ is solid whenever G is a locally compact group in class $\mathcal{S}$. In particular, if $L(G)$ is also a non-amenable factor, then $L(G)$ is prime. Combining Theorem B with the unique prime factorization results of Houdayer and Isono in [HI17] along with the generalization [AHHM18, Application 4] by Ando, Haagerup, Houdayer, and Marrakchi, we were able to obtain the following unique prime factorization result for (tensor products of) such group von Neumann algebras.

Theorem F. *Let $G = G_1 \times \cdots \times G_m$ be a direct product of locally compact groups in class $\mathcal{S}$ whose group von Neumann algebras $L(G_i)$ are nonamenable factors. If*

$$L(G) \cong N_1 \mathbin{\overline{\otimes}} \dots \mathbin{\overline{\otimes}} N_n$$

for some non-type I factors N_i, then $n \leqslant m$. Moreover, all factors N_i are prime if and only if $n = m$ and in that case (after relabeling) $L(G_i)$ is stably isomorphic to N_i for $i = 1, \dots, n$.

We prove this theorem by proving that for groups G in class $\mathcal{S}$, the group von Neumann algebra $L(G)$ belongs to the class $\mathcal{C}_{(AO)}$ introduced in [HI17].

It is worthwhile to note that for many locally compact groups G, the group von Neumann algebra $L(G)$ is amenable or even type I. For instance, the group von Neumann algebra of a connected lcsc group is always amenable by [Con76, Corollary 6.9]. However, the following group G due to Suzuki provides an example of a locally compact group whose group von Neumann algebra $L(G)$ is a non-amenable type II_∞ factor.

Example G (Suzuki)**.** Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on $\mathbb{F}_2$ by flipping the generators. Then the compact group $K = \prod_{k\in\mathbb{N}} \mathbb{Z}_2$ acts on the infinite free product $H = *_{k\in\mathbb{N}} \mathbb{F}_2$ by letting the k^{th} component of K flip the generators in the k^{th} component of H. The semi-direct product $G = H \rtimes K$ satisfies the conditions of [Suz16, Proposition] with $K_n = \prod_{k=n+1}^{\infty} \mathbb{Z}_2$ and $L_n = (*_{k=0}^{n} \mathbb{F}_2) \rtimes K$. Hence, by [Suz16, section on group von Neumann algebras], its group von Neumann algebra is a non-amenable factor of type II_∞. Moreover, G belongs to class $\mathcal{S}$ since the cocompact subgroup H does (see [BDV18, Lemma 7.2]).

Furthermore, certain classes of groups acting on trees have non-amenable group von Neumann algebras by [HR19, Theorem C and D]. Also, [Rau19b, Theorem E and F] would provide conditions on such a group G under which $L(G)$ would be a non-amenable factor. In particular, for every $q \in \mathbb{Q}$ with $0 < q < 1$ [Rau19b, Theorem G] would provide examples of groups in class $\mathcal{S}$ for which the group von Neumann algebra would be a non-amenable factor of type III_q. However, due to a mistake in [Rau19b, Lemma 5.1], there is a gap in the proofs of these results (see also [Rau19a, p 20]), and it is currently not completely clear whether these results hold as stated there.

2 Preliminaries and notation

Throughout this article, we assume all groups to be locally compact and second countable. We denote by λ_G the left Haar measure on such a group G. All topological spaces are assumed to be locally compact and Hausdorff. All actions $G \curvearrowright X$ are assumed to be continuous.

Let X be a locally compact space. We denote by $M(X)$ the space of complex Radon measures on X. We equip this space with the norm of total variation, or with the weak* topology when viewing it as the dual space of $C_0(X)$. The Borel structure from both topologies agree. We denote by $M(X)^+$ the space of positive Radon measures and $\mathrm{Prob}(G)$ the space of Radon probability measures. If a group G acts on X, then for $g \in G$ and $\mu \in M(X)$ we denote by $g \cdot \mu$ the measure defined by $(g \cdot \mu)(E) = \mu(g^{-1}E)$ for all Borel sets $E \subseteq X$.

2.1 Topological amenability

We recall from [Ana02] the notion of topological amenability for actions of locally compact groups.

Definition 2.1. Let G be a lcsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. We say that $G \curvearrowright X$ is *(topologically) amenable* if there exists a net of weakly* continuous maps $\mu_i : X \to \mathrm{Prob}(G)$ satisfying

$$\lim_i \|g \cdot \mu_i(x) - \mu_i(gx)\| = 0 \tag{2.1}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

By [Ana02, Proposition 2.2], we have the following equivalent characterization.

Proposition 2.2. *Let G be a lcsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, the following are equivalent*

(i) *$G \curvearrowright X$ is amenable*

(ii) *There exists a net $(f_i)_i$ in $C_c(X \times G)^+$ satisfying $\lim_i \int_G f_i(x, s)\,\mathrm{d}s = 1$ uniformly on compact sets for $x \in X$ and*

$$\lim_i \int_G |f_i(x, g^{-1}s) - f_i(gx, s)|\,\mathrm{d}s = 0 \tag{2.2}$$

uniformly on compact sets for $x \in X$ and $g \in G$.

Remark 2.3. Obviously, when X is σ-compact, we can replace nets by sequences in the above definition and proposition.

Remark 2.4. If X is compact, then we can take a sequence $(f_n)_n$ in $C_c(X \times G)^+$ satisfying (2.2) and such that $\int_G f_n(x, s)\,\mathrm{d}s = 1$ for every $x \in X$ and every $n \in \mathbb{N}$.

The following result shows that if X is a σ-compact space, then one can assume that the convergence in (2.1) is uniform on the whole space X, instead of only uniform on compact sets of X.

Proposition 2.5. *Let G be a lcsc group, X a σ-compact space and $G \curvearrowright X$ a continuous action. The action $G \curvearrowright X$ is amenable if and only if there exists a sequence of weakly* continuous maps $\mu_n : X \to \mathrm{Prob}(G)$ satisfying*

$$\lim_{n\to\infty} \|g \cdot \mu_n(x) - \mu_n(gx)\| = 0$$

uniformly for $x \in X$ and uniformly on compact sets for $g \in G$.

Proof. Suppose that $G \curvearrowright X$ is amenable. Take an arbitrary compact set $K \subseteq G$ and an $\varepsilon > 0$. It suffices to construct a weakly* continuous map $\mu : X \to \mathrm{Prob}(G)$ satisfying

$$\|g \cdot \mu(x) - \mu(gx)\| < \varepsilon \tag{2.3}$$

for all $g \in K$ and all $x \in X$.

Without loss of generality, we can assume that K is symmetric. Take an increasing sequence $(L_n)_{n\geqslant 1}$ of compact subsets in X such that $X = \bigcup_n L_n$. After inductively enlarging L_n, we can assume that $L_n \subseteq \mathrm{int}(L_{n+1})$ and $gL_n \subseteq L_{n+1}$ for every $g \in K$. Using the amenability of $G \curvearrowright X$, we can take a sequence of weakly* continuous maps $\nu_n : X \to \mathrm{Prob}(G)$ satisfying $\|g \cdot \nu_n(x) - \nu_n(gx)\| < 2^{-n}$ for all $g \in K$, $x \in L_n$ and $n \in \mathbb{N} \setminus \{0\}$. Set $L_n = \emptyset$ for $n \leqslant 0$. Fix $n \geqslant 1$ such that $14/n < \varepsilon$ and take continuous functions $f_k : X \to [0, 1]$ such that $f_k(x) = 1$ whenever $x \in L_k \setminus L_{k-n}$ and $f_k(x) = 0$ whenever $x \in L_{k-n-1}$ or $x \in X \setminus L_{k+1}$.

For every $x \in X$, we denote $|x| = \max\{k \in \mathbb{N} \mid x \notin L_k\}$. We set

$$\tilde{\mu}(x) = \sum_{k=0}^{\infty} f_k(x)\nu_k(x) = f_{|x|}(x)\nu_{|x|}(x) + f_{|x|+n+1}(x)\nu_{|x|+n+1}(x) + \sum_{k=|x|+1}^{|x|+n} \nu_k(x).$$

for $x \in X$ and define $\mu : X \to \mathrm{Prob}(G) : x \mapsto \tilde{\mu}(x)/\|\tilde{\mu}(x)\|$. Clearly, μ is weakly* continuous. To prove that μ satisfies (2.3), fix $x \in X$ and $g \in K$. Since $gL_k \subseteq L_{k+1}$ and $g^{-1}L_k \subseteq L_{k+1}$ for every $k \in \mathbb{N}$, we have $|x| - 1 \leqslant |gx| \leqslant |x| + 1$ and hence

$$\|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leqslant 6 + \sum_{k=|x|+1}^{|x|+n} \|g \cdot \nu_k(x) - \nu_k(gx)\| \leqslant 7,$$

where we used that $g \in K$ and $x \in L_k$ for $k = |x| + 1, \dots, |x| + n$. Hence,

$$\|g \cdot \mu(x) - \mu(gx)\| \leqslant \frac{2}{\|\tilde{\mu}(x)\|} \|g \cdot \tilde{\mu}(x) - \tilde{\mu}(gx)\| \leqslant \frac{14}{n} < \varepsilon$$

as was required. □

The following result can for instance be found in [BO08, Exercise 15.2.1] for discrete groups. The proof for locally compact groups is exactly the same.

Lemma 2.6. *Let G be a lcsc group, X a locally compact space and $G \curvearrowright X$ a continuous action. Then, $G \curvearrowright X$ is amenable if and only if the induced action $G \curvearrowright \mathrm{Prob}(X)$ is amenable, where $\mathrm{Prob}(X)$ is equipped with the weak* topology.*

2.2 Exactness

The following definition of exactness was given by Kirchberg and Wassermann in [KW99a]. Recall that a G-C*-algebra is a C*-algebra A together with a $\|.\|$-continuous action $G \curvearrowright A$ by *-isomorphisms. We denote by $A \rtimes_r G$ the reduced crossed product.

Definition 2.7. A lcsc group G is called *exact* if for every G-equivariant exact sequence of G-C*-algebras

$$0 \to A \to B \to C \to 0,$$

the induced sequence

$$0 \to A \rtimes_r G \to B \rtimes_r G \to C \rtimes_r G \to 0$$

is exact.

It is an immediate consequence of this definition that the reduced group C*-algebra $C^*_r(G)$ is exact whenever G is exact. The converse is also true for discrete groups (see [KW99a, Theorem 5.2]), but is still open for locally compact groups. The class of exact groups is very large and contains among others all (weakly) amenable groups [HK94; BCL17], linear groups [GHW05] and hyperbolic groups [Ada94]. Examples of non-exact groups were given by Gromov [Gro03; AD08] and Osajda [Osa14].

As before, we denote by $\beta^{lu}G$ the spectrum of the algebra $C_b^{lu}(G)$ of bounded left-uniformly continuous functions on G. By [Ana02, Theorem 7.2] and [BCL17, Theorem A] we have that a group G is exact if and only if G admits an amenable action on some compact space, or equivalently if the action $G \curvearrowright \beta^{lu}G$ induced by left translation is amenable.

3 Class $\mathcal{S}$ and boundary actions small at infinity

The main goal of this section is to prove Theorems B and C, but we first need the following equivalent characterizations of the existence of a map satisfying (1.1). Note that point (i) in the proposition below is property (S) in the sense of [BDV18].

As before, we denote by $\mathcal{S}(G)$ the space $\left\{f \in L^1(G)^+ \;\middle|\; \|f\|_1 = 1\right\}$ of probability measures on G that are absolutely continuous with respect to the Haar measure. There is an obvious G-equivariant norm-preserving embedding $\mathcal{S}(G) \hookrightarrow \mathrm{Prob}(G)$.

Proposition 3.1. *Let G be a lcsc group. Then, the following are equivalent.*

(i) *There is a $\|.\|_1$-continuous map $\eta : G \to \mathcal{S}(G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0$$

uniformly on compact sets for $g, h \in G$.

(ii) *There exists a $\|.\|$-continuous map $\eta : G \to \mathrm{Prob}(G)$ satisfying*

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

(iii) *There exists a sequence of Borel maps $\eta_n : G \to M(G)^+$ satisfying*

$$\liminf_{n\to\infty} \liminf_{k\to\infty} \|\eta_n(k)\| > 0$$

and

$$\lim_{n\to\infty} \limsup_{k\to\infty} \sup_{g,h\in K} \|\eta_n(gkh) - g \cdot \eta_n(k)\| = 0$$

for all compact sets $K \subseteq G$.

Proof. The implications (i)⇒(ii)⇒(iii) are trivial. We prove the reverse implications (iii)⇒(ii)⇒(i).

First, we prove (ii)⇒(i). The proof follows the lines of [Ana02, Proposition 2.2]. Let $\eta : G \to \mathrm{Prob}(G)$ be as in (ii). We construct $\tilde\eta : G \to \mathcal{S}(G)$ as follows. Take an $f \in C_c(G)^+$ with $\int_G f(t)\,\mathrm{d}t = 1$. Define

$$\tilde\eta(g)(s) = \int_G f(t^{-1}s)\,\mathrm{d}\eta(g)(t)$$

for $s, g \in G$. A similar calculation as in [Ana02, Proposition 2.2] checks that $\tilde\eta(g) \in \mathcal{S}(G)$ for every $g \in G$, that $\tilde\eta$ is $\|.\|_1$-continuous and that $\|\tilde\eta(gkh) - g \cdot \tilde\eta(k)\|_1 \to 0$ uniformly on compact sets for $g, h \in G$ whenever $k \to \infty$.

The implication (iii)⇒(ii) follows from the technical lemmas 3.2 and 3.4 below applied on the spaces $X = Y = H = G$ with the actions $G \times G \curvearrowright X$ and $G \curvearrowright Y$ by $(g, k) \cdot x = gxk^{-1}$ and $(g, h) \cdot y = gy$ for $g, k \in G$, $x \in X$ and $y \in Y$. □

The following is a more abstract and slightly more general version of the trick in [BO08, Exercise 15.1.1]. It will be used several times in this article.

Lemma 3.2. *Let X and Y be σ-compact spaces and G a lcsc group. Suppose that $G \curvearrowright X$ and $G \curvearrowright Y$ are continuous actions. If there exists a sequence of Borel maps $\eta_n : X \to M(Y)^+$ satisfying*

$$\lim_{n\to\infty} \limsup_{x\to\infty} \sup_{g\in K} \|\eta_n(gx) - g\cdot\eta_n(x)\| = 0 \tag{3.1}$$

for all compact sets $K \subseteq G$ and

$$\liminf_{n\to\infty} \liminf_{x\to\infty} \|\eta_n(x)\| > 0.$$

Then, there exists a Borel map $\eta : X \to \mathrm{Prob}(Y)$ such that

$$\lim_{x\to\infty} \|\eta(gx) - g\cdot\eta(x)\| = 0 \tag{3.2}$$

uniformly on compact sets for $g \in G$. Moreover, if the maps η_n are assumed to be $\|.\|$-continuous, then also η can be assumed to be $\|.\|$-continuous. If the maps η_n are weakly continuous and the maps $x \mapsto \|\eta_n(x)\|$ are continuous for every $n \in \mathbb{N}$, then also η can be assume to be weakly* continuous.*

Proof. After passing to a subsequence and replacing values of η_n on compact sets, we can assume that there exists a $\delta > 0$ such that $\|\eta_n(x)\| \geqslant \delta$ for all $n \in \mathbb{N}$ and all $x \in X$. Set $\tilde\eta_n(x) = \eta_n(x)/\|\eta_n(x)\|$ for all $x \in X$. Note that $(\tilde\eta_n)_n$ still satisfies (3.1) for all compact sets $K \subseteq G$. Moreover, the maps $\tilde\eta_n$ are $\|.\|$-continuous whenever η_n is $\|.\|$-continuous. If the maps η_n are weakly* continuous and the maps $x \mapsto \|\eta_n(x)\|$ are continuous, then also the maps $\tilde\eta_n$ are weakly* continuous.

Take an increasing sequence $(K_n)_n$ of compact symmetric neighbourhoods of the unit e in G such that $G = \bigcup_n \mathrm{int}(K_n)$. After passing to a subsequence of $(\tilde\eta_n)_n$, we find compact sets $L_n \subseteq X$ such that

$$\|\tilde\eta_n(gx) - g\cdot\tilde\eta_n(x)\| \leqslant 2^{-n+1}$$

for all $g \in K_n$ and $x \in X \setminus L_n$. After inductively enlarging L_n, we can assume that the sequence $(L_n)_n$ is increasing, that $L_n \subseteq \mathrm{int}(L_{n+1})$, that $gL_n \subseteq L_{n+1}$ for all $g \in K_n$ and that $X = \bigcup_n L_n$. Moreover, we can also assume that $L_0 = \emptyset$.

For every $x \in X$, we denote $|x| = \max\{n \in \mathbb{N} \mid x \notin L_n\}$. Furthermore, we denote $h(n) = \lfloor n/2 \rfloor + 1$. For all $n \geqslant 1$, we take a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ if $x \in L_{2n} \setminus L_n$ and $f_n(x) = 0$ if $x \in L_{n-1}$ or $x \in X \setminus L_{2n+1}$. Take $f_0 : X \to [0,1]$ such that $f_0(x) = 1$ if $x \in L_1$ and $f_0(x) = 0$ if $x \in X \setminus L_2$. For $x \in X$, we set

$$\begin{aligned}\mu(x) &= \sum_{k=0}^{+\infty} f_k(x)\tilde\eta_k(x)\\ &= f_{h(|x|)-1}(x)\,\tilde\eta_{h(|x|)-1}(x) + f_{|x|+1}(x)\,\tilde\eta_{|x|+1}(x) + \sum_{k=h(|x|)}^{|x|} \tilde\eta_k(x).\end{aligned}$$

Now, define $\eta : X \to \mathrm{Prob}(Y)$ by $\eta(x) = \mu(x)/\|\mu(x)\|$. Note that η is $\|.\|$-continuous (resp. weakly* continuous) whenever the maps $\tilde\eta_n$ are.

To prove that η satisfies (3.2), take an $\varepsilon > 0$ and a compact subset $K \subseteq G$. Take $n_0 \geqslant 1$ such that $K \subseteq K_{n_0}$ and take $n_1 > \max\{2n_0, 32/\varepsilon\}$. We claim that $\|\eta(g\cdot x) - g\cdot\eta(x)\| < \varepsilon$ whenever $x \in G \setminus L_{n_1}$ and $g \in K$. Indeed, fix $g \in K$ and $x \in G \setminus L_{n_1}$. Take $n = |x|$. Since $gL_{n+1} \subseteq L_{n+2}$ and $g^{-1}L_{n-1} \subseteq L_n$, we have that $n - 1 \leqslant |gx| \leqslant n + 1$. Hence,

$$\|\mu(gx) - g\cdot\mu(x)\| \leqslant 6 + \sum_{k=h(n)}^{n} \|\tilde\eta_k(gx) - g\cdot\tilde\eta_k(x)\| \leqslant 8,$$

since $g \in K_k$ and $x \in X \setminus L_k$ for $k = h(n), \dots, n$. We conclude that

$$\|\eta(gx) - g\cdot\eta(x)\| \leqslant \frac{2}{\|\mu(x)\|}\,\|\mu(gx) - g\cdot\mu(x)\| \leqslant \frac{4}{n}\cdot 8 < \varepsilon$$

which proves the claim. ∎

Remark 3.3. Using almost exactly the same proof as above, one can actually prove the following slightly more general result: suppose that for every $\varepsilon > 0$ and every compact set $K \subseteq G$, there exists a compact set $L \subseteq X$ such that for all compact sets $L' \subseteq X$, there exists a map $\eta' : X \to M(Y)^+$ such that

$$\frac{\|\eta'(gx) - g \cdot \eta'(x)\|}{\|\eta'(x)\|} < \varepsilon \tag{3.3}$$

whenever $g \in K$ and $x \in L' \setminus L$. Then, there exists a map $\eta : X \to \text{Prob}(Y)$ as in (3.2). Indeed, using the notation of the proof, we can take the compact sets $L_n \subseteq X$ and the maps $\eta_n : X \to M(Y)^+$ such that $\|\tilde{\eta}_n(gx) - g \cdot \tilde{\eta}_n(x)\| < 2^{-n+1}$ for all $g \in K_n$ and $x \in L_{2n} \setminus L_n$, where again $\tilde{\eta}_n(x) = \eta_n(x)/\|\eta_n(x)\|$. The rest of the proof holds verbatim.

The following lemma will be used several times to replace Borel maps by continuous maps.

Lemma 3.4. *Let H and G be lcsc groups and Y a locally compact space. Suppose that $G \curvearrowright^{\alpha} H$ and $G \curvearrowright Y$ are arbitrary continuous actions. If there exists a Borel map $\eta : H \to \text{Prob}(Y)$ satisfying*

$$\lim_{h\to\infty} \big\|\eta\big(\alpha_g(h)\big) - g \cdot \eta(h)\big\| = 0 \qquad \text{and} \qquad \lim_{h\to\infty} \big\|\eta\big(\alpha_g(h)k\big) - \eta\big(\alpha_g(hk)\big)\big\| = 0$$

uniformly on compact sets for $g \in G$ and $k \in H$, then there exists a $\|.\|$-continuous map $\tilde{\eta} : H \to \text{Prob}(Y)$ map satisfying

$$\lim_{h\to\infty} \big\|\tilde{\eta}\big(\alpha_g(h)\big) - g \cdot \tilde{\eta}(h)\big\| = 0$$

uniformly on compact sets for $g \in G$.

Proof. Fix a compact neighbourhood K of the unit e in H with $\lambda_H(K) = 1$. We define $\tilde{\eta} : H \to \text{Prob}(Y)$ by

$$\tilde{\eta}(g) = \int_K \eta(gk)\, \mathrm{d}k.$$

One check that the map $\tilde{\eta}$ satisfies the conclusions of the lemma. □

We are now ready to prove Theorem B.

Proof of Theorem B. First, we prove (i)⇒(ii). Let $\eta : G \to \text{Prob}(G)$ be a map as in the definition of class $\mathcal{S}$. Consider the u.c.p. map $\eta_* : C_b^{lu}(G) \to C(h^u G)$ defined by

$$(\eta_* f)(g) = \int_G f(s)\, \mathrm{d}\eta(g)(s)$$

for $f \in C_b^{lu}(G)$ and $g \in G$. A straightforward calculation checks that indeed $\eta_*(f) \in C(h^u G) \subseteq C_b^u(G)$ for every $f \in C_b^{lu}(G)$. Moreover, $\eta_*(\lambda_g f) - \lambda_g(\eta_* f) \in C_0(G)$ for all $f \in C_b^{lu}(G)$. Hence, composing with the quotient map $\pi : C(h^u G) \to C(\nu^u G) \cong C(h^u G)/C_0(G)$ yields a G-equivariant u.c.p. map $\pi \circ \eta_* : C_b^{lu}(G) \to C(\nu^u G)$. By dualization, we obtain a weakly* continuous G-equivariant map $\nu^u G \to \text{Prob}(\beta^{lu} G)$ given by $x \mapsto \delta_x \circ \pi \circ \eta_*$. Since G is exact, the action $G \curvearrowright \beta^{lu} G$ is amenable and hence so is $G \curvearrowright \text{Prob}(\beta^{lu} G)$ (see Lemma 2.6). Composing with the G-equivariant map $\nu^u G \to \text{Prob}(\beta^{lu} G)$ above, yields that $G \curvearrowright \nu^u G$ is amenable.

Now, we prove (ii)⇔(iii). The implication from right to left is trivial. To prove the other implication, take an arbitrary compact subset $K \subseteq G$ and an $\varepsilon > 0$. By Proposition 2.2, it suffices to construct a function $h \in C_c(h^u G \times G)^+$ such that $\int_G h(x, s)\, \mathrm{d}s = 1$ for every $x \in h^u G$ and

$$\int_G |h(x, g^{-1}s) - h(gx, s)|\, \mathrm{d}s < \varepsilon \tag{3.4}$$

for all $x \in h^u G$ and $g \in K$.

By Proposition 2.2 and Remark 2.4, we find a function $f \in C_c(\nu^u G \times G)^+$ with $\int_G f(x, s)\, \mathrm{d}s = 1$ such that $\int_G |f(x, g^{-1}s) - f(gx, s)|\, \mathrm{d}s < \varepsilon/2$ for all $x \in \nu^u G$ and $g \in K$. By the Tietze Extension Theorem, we can extend f to an $\tilde{f} \in C_c(h^u G \times G)^+$. Since

$$\limsup_i \int_G |\tilde{f}(x_i, g^{-1}s) - \tilde{f}(gx_i, s)|\, \mathrm{d}s \leqslant \sup_{x\in\nu^u G} \sup_{g\in K} \int_G |f(x, g^{-1}s) - f(gx, s)|\, \mathrm{d}s < \frac{\varepsilon}{2},$$

for every net $(x_i)_i$ in G converging to an $x \in \nu^u G$, we can take a compact set $L \subseteq G$ such that

$$\int_G |\tilde{f}(x, g^{-1}s) - \tilde{f}(gx, s)| \, \mathrm{d}s < \frac{\varepsilon}{2}.$$

for all $x \in h^u G \setminus L$ and $g \in K$. After possibly enlarging L and renormalizing $\tilde{f}$, we can moreover assume that $\int_G \tilde{f}(x, s) \, \mathrm{d}s = 1$.

Fix a function $a \in C_c(G)^+$ with $\int_G a(s) \, \mathrm{d}s = 1$. Using Lemma 3.5 below, we can take a function $\zeta \in C_c(G)^+$ such that $\zeta|_L = 1$ and $|\zeta(gh) - \zeta(h)| < \varepsilon/4$ for $h \in G$ and $g \in K$. Now, define $h \in C_c(h^u G \times G)$ by

$$h(x, s) = \begin{cases} \zeta(x) a(x^{-1}s) + \big(1 - \zeta(x)\big) \tilde{f}(x, s) & \text{if } x \in G, \\ \tilde{f}(x, s) & \text{if } x \in \nu^u G. \end{cases}$$

A straightforward calculation shows that h satisfies (3.4).

Next, we prove (ii)⇒(iv) Denote by X the spectrum of $A = C_b^u(G)/C_0(G)$. Since $C(h^u G) \subseteq C_b^u(G)$, we have a natural embedding $C(\nu^u G) \hookrightarrow A$, which in turn induces a continuous map $\varphi_\ell : X \to \nu^u G$. Note that φ_ℓ is $G \times G$-equivariant with respect to the actions induced by left and right translation. Similarly, we get a $G \times G$-equivariant map $\varphi_r : X \to \nu_r^u G$, where $\nu_r^u G$ denotes the spectrum of the algebra

$$C(\nu_r^u G) = \{ f \in C_b^u(G) \mid \lambda_g f - f \in C_0(G) \}$$

and the action $G \times G \curvearrowright \nu_r^u G$ is induced by left and right translation. By assumption, the action $G \times 1 \curvearrowright \nu^u G$ is amenable, and by symmetry so is $1 \times G \curvearrowright \nu_r^u G$. Hence, the diagonal action $G \times G \curvearrowright \nu^u G \times \nu_r^u G$ is also amenable. Now, the conclusion follows from the $G \times G$-equivariance of the map $\varphi_\ell \times \varphi_r : X \to \nu^u G \times \nu_r^u G$.

Finally, we prove (iv)⇒(i). By [Ana02, Theorem 7.2], the group G is exact. Denote again by X the spectrum of $A = C_b^u(G)/C_0(G)$. Denoting by $\beta^u G$ the spectrum of $C_b^u(G)$, we get $X = \beta^u G \setminus G$. By Proposition 2.2 and Remark 2.4, we can take a sequence $(f_n)_n$ of functions in $C_c(X \times G \times G)^+$ such that $\int_{G \times G} f_n(x, s, t) \, \mathrm{d}s \, \mathrm{d}t = 1$ for all $x \in X$ and $n \in \mathbb{N}$, and such that

$$\lim_{n \to \infty} \int_{G \times G} |f_n(x, g^{-1}s, h^{-1}t) - f_n\big((g, h) \cdot x, s, t\big)| \, \mathrm{d}s \, \mathrm{d}t = 0 \tag{3.5}$$

uniformly for $x \in X$ and uniformly on compact sets for $g, h \in G$. As before, the Tietze Extension Theorem yields extensions $\tilde{f}_n \in C_c(\beta^u G \times G \times G)^+$ of each f_n. For each $x \in \beta^u G$ and $n \in \mathbb{N}$, we define $\eta_n(x) \in M(G)^+$ as the measure with density function $s \mapsto \int_G \tilde{f}_n(x, s, t) \, \mathrm{d}t$ with respect to the Haar measure on G. This yields $\|.\|$-continuous maps $\eta_n : \beta^u G \to M(G)^+$. By (3.5), the restrictions of η_n to $G \subseteq \beta^u G$ satisfy the conditions of Proposition 3.1 (iii). ☐

In the proof above, we used the following easy lemma.

Lemma 3.5. *Let G be a lcsc group. For all compact subsets $K, L \subseteq G$ and all $\varepsilon > 0$, there exists a continuous function $f \in C_c(G)$ satisfying $f|_L = 1$ and*

$$|f(kgk') - f(g)| < \varepsilon$$

for $k, k' \in K$ and $g \in G$.

Proof. We can assume that K is symmetric and that $e \in \mathrm{int}(K)$. Take continuous functions $f_n : G \to [0, 1]$ such that $f_n(g) = 1$ for $g \in K^n L K^n$ and $\operatorname{supp} f_n \subseteq K^{n+1} L K^{n+1}$. Take $N \in \mathbb{N}$ such that $1/N < \varepsilon/4$ and set $f = \frac{1}{N} \sum_{k=0}^{N-1} f_n$. ☐

We end this section by proving Theorem C.

Proof of Theorem C. The implication from left to right is trivial. To prove the converse implication, note that by exactness of G and Lemma 2.6, the action $G \curvearrowright \mathrm{Prob}(\beta^{lu} G)$ is amenable. Take a sequence $\theta_n : \mathrm{Prob}(\beta^{lu} G) \to \mathrm{Prob}(G)$ such that

$$\lim_{n \to \infty} \|\theta_n(g \cdot \mu) - g \cdot \theta_n(\mu)\| = 0$$

uniformly for $\mu \in \text{Prob}(\beta^{lu}G)$ and uniformly on compact sets for $g \in G$. Now, for the composition $\eta_n = \theta_n \circ \eta$ we get

$$\begin{aligned}\|\eta_n(gkh) - g \cdot \eta_n(k)\| &\leqslant \|\eta(gkh) - g \cdot \eta(k)\| + \big\|\theta_n\big(g \cdot \eta(k)\big) - g \cdot \theta_n\big(\eta(k)\big)\big\| \\ &\leqslant \|\eta(gkh) - g \cdot \eta(k)\| + \sup_{\mu \in \text{Prob}(\beta^{lu}G)} \big\|\theta_n(g \cdot \mu) - g \cdot \theta_n(\mu)\big\|\end{aligned}$$

whenever $g, h, k \in G$. It follows that $(\eta_n)_n$ satisfies the conditions of Proposition 3.1 (iii). ☐

4 Locally compact wreath products in class $\mathcal{S}$

In this section, we prove Theorem D. Before we start the proof of this result, we need a few preliminary results. The first is a locally compact version of [BO08, Lemma 15.2.6]. This result can be proven in a similar way as in [BO08, Lemma 15.2.6]. However, we provide a different proof, not requiring exactness.

Proposition 4.1. *Let G be an lcsc group and K a closed, amenable subgroup. If there exists a Borel map $\eta : G \to \text{Prob}(G/K)$ such that*

$$\lim_{k \to \infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$. Then, G has property (S), i.e. there exists a $\|.\|$-continuous map $\tilde{\eta} : G \to \text{Prob}(G)$ satisfying (1.1).

Proof. Using Lemma 3.4, we can assume that η is $\|.\|$-continuous. The proof then follows easily from Lemma 4.3 below. ☐

Let G be a group and $H \subseteq G$ a closed subgroup. Denote by $p : G \to G/H$ the quotient map. Let $\sigma : G/H \to G$ be a locally bounded Borel cross section for p, i.e. a Borel map satisfying $p \circ \sigma = \text{Id}_{G/H}$ that maps compact sets onto precompact sets (see [Mac52, Lemma 1.1]). We identify G with $G/H \times H$ via the map

$$\phi : G \to G/H \times H : g \mapsto \big(gH, \sigma(gH)^{-1}g\big). \tag{4.1}$$

Under this identification the action by left translation is given by $k \cdot (gH, h) = \big(kgH, \omega(k, gH)h\big)$, where $\omega(k, gH) = \sigma(kgH)^{-1}k\sigma(gH)$.

The identification map ϕ is not continuous, but it is bi-measurable and maps (pre)compact sets to precompact sets. This allows us to identify the spaces $\text{Prob}(G)$ and $\text{Prob}(G/H \times H)$ via the map $\mu \mapsto \phi_*\mu$. Note that this identification map is continuous with respect to the norm topology on both spaces, but not with respect to the weak* topology on both spaces. We use the above identifications in the following two lemmas.

Lemma 4.2. *Let G be a lcsc group and $H \subseteq G$ a closed, amenable subgroup. Let $(\nu_n)_n$ be a sequence in $\text{Prob}(H)$ such that $\|h \cdot \nu_n - \nu_n\| \to 0$ uniformly on compact sets for $h \in H$ whenever $n \to \infty$. Then,*

$$\lim_{n \to \infty} \|h \cdot (\mu \otimes \nu_n) - (h \cdot \mu) \otimes \nu_n\| = 0$$

uniformly on compact sets for $g \in G$ and uniformly on weakly compact sets for $\mu \in \text{Prob}(G/H)$.*

Proof. Fix compact subsets $K \subseteq G$ and $\mathcal{L} \subseteq \text{Prob}(G/H)$, and take $\varepsilon > 0$. A straightforward calculation yields

$$\|k \cdot (\mu \otimes \nu_n) - (k \cdot \mu) \otimes \nu_n\| \leqslant \int_{G/H} \|\omega(k, gH) \cdot \nu_n - \nu_n\| \, \text{d}\mu(gH)$$

for all $\mu \in \text{Prob}(G/H)$, all $k \in G$ and all $n \in \mathbb{N}$. Take a compact set $L \subseteq G/H$ such that $\mu(L) > 1 - \varepsilon$ for all $\mu \in \mathcal{L}$. Since ω maps compact sets to precompact sets, we find an $n_0 \in \mathbb{N}$ such that $\|\omega(k, gH) \cdot \nu_n - \nu_n\| < \varepsilon$ for all $n \geqslant n_0$, all $k \in K$ and all $gH \in L$. Then, $\|k \cdot (\mu \otimes \nu_n) - (k \cdot \mu) \otimes \nu_n\| \leqslant 3\varepsilon$ for $n \geqslant n_0$, $k \in K$ and $\mu \in \mathcal{L}$, thus proving the result. ☐

Lemma 4.3. *Let G and H be lcsc groups, $\pi : G \to H$ a continuous morphism and $K \subseteq H$ a closed, amenable subgroup. Let $G \curvearrowright X$ be a continuous action on some σ-compact space X. Let $G \curvearrowright \mathrm{Prob}(H)$ (resp. $G \curvearrowright \mathrm{Prob}(H/K)$) be defined by $g \cdot \mu = \pi(g) \cdot \mu$ for $g \in G$ and $\mu \in \mathrm{Prob}(H)$ (resp. $\mu \in \mathrm{Prob}(H/K)$). If there exists a weakly* continuous map $\eta : X \to \mathrm{Prob}(H/K)$ such that*

$$\lim_{x\to\infty} \|\eta(gx) - g \cdot \eta(x)\| = 0$$

uniformly on compact sets for $g \in G$. Then, there exists a Borel map $\tilde{\eta} : X \to \mathrm{Prob}(H)$ such that

$$\lim_{x\to\infty} \|\tilde{\eta}(gx) - g \cdot \tilde{\eta}(x)\| = 0$$

uniformly on compact sets for $g \in G$. Moreover, if η is assumed to be $\|.\|$-continuous then $\tilde{\eta}$ can also be assumed to be $\|.\|$-continuous.

Proof. Fix a locally bounded Borel cross section $\sigma : H/K \to H$ for the quotient map $p : H \to H/K$, and identify H with $H/K \times K$ and $\mathrm{Prob}(H)$ with $\mathrm{Prob}(H/K \times K)$ as in (4.1).

Take a sequence $(\nu_n)_n$ in $\mathrm{Prob}(K)$ such that $\|k \cdot \nu_n - \nu_n\| \to 0$ uniformly on compact sets for $k \in K$ whenever $n \to \infty$. Using Lemma 4.2, we construct maps as in Remark 3.3 as follows. Fix an $\varepsilon > 0$ and a compact $C \subseteq G$. Take a compact $L \subseteq X$ such that $\|\eta(gx) - g \cdot \eta(x)\| < \varepsilon$ for all $g \in C$ and $x \in X \setminus L$. Fix any compact set $L' \subseteq X$. Applying Lemma 4.2 to the weak* compact set $\eta(L')$, we find an $n \in \mathbb{N}$ such that $\big\|\big(g \cdot \eta(x)\big) \otimes \nu_n - g \cdot \big(\eta(x) \otimes \nu_n\big)\big\| < \varepsilon$ for any $x \in L'$ and $g \in C$. Hence, $\big\|\eta(gx) \otimes \nu_n - g \cdot \big(\eta(x) \otimes \nu_n\big)\big\| \leqslant 2\varepsilon$ for any $g \in C$ and any $x \in L' \setminus L$. We conclude that the map $\eta' : X \to \mathrm{Prob}(H)$ defined by $\eta'(x) = \eta(x) \otimes \nu_n$ is as in (3.3). Moreover, if η is $\|.\|$-continuous, then so is μ. ☐

The second result that we need before proving Theorem D characterizes when a semi-direct product belongs to class $\mathcal{S}$. By definition $G = B \rtimes H$ belongs to class $\mathcal{S}$ whenever it is exact and there exists a map $\eta : G \to \mathrm{Prob}(G)$ satisfying $\big\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \mu(b,h)\big\| \to 0$ uniformly on compact sets for $(a,k),(a',k') \in G$ whenever $(b,h) \to \infty$. The result below shows that is suffices that there exist *two* such maps one of which satisfies the convergence above when $b \to \infty$ and the other when $h \to \infty$.

Proposition 4.4. *Let $G = B \rtimes_\alpha H$ be a semi-direct product of lcsc groups. Then, G is in class $\mathcal{S}$ if and only if B and H are exact, and there exists Borel maps $\mu : G \to \mathrm{Prob}(G)$ and $\nu : G \to \mathrm{Prob}(G)$ such that*

$$\lim_{b\to\infty} \big\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \mu(b,h)\big\| = 0 \tag{4.2}$$

uniformly on compact sets for $a, a' \in B$ and $k, h, k' \in H$, and such that

$$\lim_{h\to\infty} \big\|\nu\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \nu(b,k)\big\| \tag{4.3}$$

uniformly for $b \in B$ and uniformly on compact sets for $a, a' \in B$ and $k, k' \in H$.

Proof. The only if part is immediate. To prove the converse, note first that G is exact as an extension of an exact group by an exact group (see [KW99b, Theorem 5.1]). Fix a compact set $K \subseteq G$ and an $\varepsilon > 0$. By Proposition 3.1 (iii), it suffices to find a Borel map $\eta : G \to \mathrm{Prob}(G)$ and a compact set $L \subseteq G$ such that

$$\big\|\eta\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \eta(b,h)\big\| < \varepsilon \tag{4.4}$$

for all $(a,k),(a',k') \in K$ and all $(b,h) \in G \setminus L$.

Since K is compact, we can take compact subsets $K_B \subseteq B$ and $K_H \subseteq H$ such that

$$K \subseteq \{(b,h) \mid b \in K_B, h \in K_H\}\,.$$

By assumption, we can take a compact set $\widetilde{L}_H \subseteq H$ such that $\big\|\nu\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \nu(b,k)\big\| < \varepsilon/2$ whenever $a, a' \in K_B$, $b \in B$, $k, k' \in K_H$ and $h \in H \setminus \widetilde{L}_H$.

Using Lemma 3.5, we take a function $f \in C_c(H)$ such that $f(h) = 1$ for $h \in \widetilde{L}_H$ and $|f(khk') - f(h)| < \varepsilon/4$ whenever $h \in H$ and $k, k' \in K_H$. Set $L_H = \operatorname{supp} f$. Take a compact set $L_B \subseteq B$ such that $\big\|\mu\big((a,k)(b,h)(a',k')\big) - (a,k) \cdot \mu(b,h)\big\| < \varepsilon/2$ whenever $a, a' \in K_B$, $b \in G \setminus L_B$, $k, k' \in K_H$ and $h \in L_H$.

Now, define $\eta : G \to \mathrm{Prob}(G)$ by

$$\eta(b,h) = f(h)\mu(b,h) + \big(1 - f(h)\big)\nu(b,h)$$

for $(b,h) \in G$. Set $L = \{(b,h) \in G \mid b \in L_B, h \in L_H\}$. Fix $(a,k),(a',k') \in K$ and $(b,k) \in G \setminus L$. Denote $g = (b,h)$, $g' = (a,k)(b,h)(a',k')$ and $g'' = (a,k)$. We have

$$\|\eta(g') - g'' \cdot \eta(g)\| \leqslant f(h) \left\|\mu\big(g'\big) - g'' \cdot \mu(g)\right\| + \big(1 - f(h)\big) \left\|\nu\big(g'\big) - g'' \cdot \nu(g)\right\| + \frac{\varepsilon}{2}$$

We are in one of the following three cases: either $h \in H \setminus L_H$, or $h \in L_H \setminus \widetilde{L}_H$ and $b \in B \setminus L_B$, or $h \in \widetilde{L}_H$ and $b \in B \setminus L_B$. In all three cases (4.4) holds. □

Remark 4.5. Note that (4.2) is equivalent with the existence of a map $\tilde{\mu} : B \to \mathrm{Prob}(G)$ satisfying

$$\lim_{b\to\infty} \|\tilde{\mu}(aba') - a \cdot \tilde{\mu}(b)\| = 0 \qquad \text{and} \qquad \lim_{b\to\infty} \left\|\tilde{\mu}\big(\alpha_h(b)\big) - h \cdot \tilde{\mu}(b)\right\| = 0$$

uniformly on compact sets for $a, a' \in B$ and $h \in H$. Indeed, the restriction of a map as in (4.2) satisfies the above equations. Conversely, given a map $\tilde{\mu}$ as above, the map $\mu : G \to \mathrm{Prob}(G)$ defined by $\mu(b,h) = \tilde{\mu}(b)$ satisfies (4.2).

When the group B is amenable, the previous result specializes to the corollary below. In the setting of countable groups, this result was proved by Ozawa in [Oza06, proof of Corollary 4.5] and [Oza09, Section 3]. However, the proof provided there does not carry over to the locally compact setting, since, as we explained in the introduction, the characterization of class $\mathcal{S}$ in terms of a u.c.p. map $\varphi : \mathrm{C}^*_r(G) \otimes_{\min} \mathrm{C}^*_r(G) \to B(L^2(G))$ satisfying $\varphi(x \otimes y) - \lambda(x)\rho(y) \in K(L^2(G))$ (see [BO08, Proposition 15.1.4]) does not hold in this setting. Also the method used in [BO08, Section 15.2] can not be applied, since for a locally compact group G the crossed product $C(X) \rtimes_r G$ can be nuclear while $G \curvearrowright X$ is not amenable.

Corollary 4.6. *Let $G = B \rtimes_\alpha H$ be a semi-direct product of lcsc groups with B amenable. Then G is in class $\mathcal{S}$ if and only if H is in class $\mathcal{S}$ and there is a Borel map $\mu : B \to \mathrm{Prob}(H)$ such that*

$$\lim_{b\to\infty} \left\|\mu\big(\alpha_h(b)\big) - h \cdot \mu(b)\right\| = 0 \qquad \text{and} \qquad \lim_{b\to\infty} \|\mu(aba') - \mu(b)\| = 0$$

uniformly on compact sets for $h \in H$ and $a, a' \in B$.

Proof. The only if part is clear. Conversely, let $\mu : B \to \mathrm{Prob}(H)$ be a map as above. Applying Lemmas 3.4 and 4.3 yields a map $\tilde{\mu} : B \to \mathrm{Prob}(G)$ as in Remark 4.5. Applying Lemma 4.3 to the map $\eta : H \to \mathrm{Prob}(H)$ from the definition of class $\mathcal{S}$ and composing it with the projection $(b,h) \mapsto h$ yields a map satisfying (4.3). □

We are now ready to prove Theorem D. The suitable notion of wreath products for locally compact groups was introduced by Cornulier in [Cor17]. Let B and H be lcsc groups, X a countable set with continuous action $H \curvearrowright X$ and $A \subseteq B$ a compact open subgroup. The *semi-restricted power* $B^{X,A}$ is defined by

$$B^{X,A} = \left\{(b_x)_{x\in X} \in B^X \;\middle|\; b_x \in A \text{ for all but finitely many } x \in X\right\}.$$

It is a lcsc space when equipped with the topology generated by the open sets $\prod_{x\in X} C_x$ where $C_x \subseteq B$ is open for every $x \in X$ and $C_x = A$ for all but finitely many $x \in X$. For $b \in B^{X,A}$, we denote $\mathrm{supp}_A\, b = \{x \in X \mid b(x) \notin A\}$.

Denote by α the action of H on $B^{X,A}$ by translation, i.e. $\alpha_h(b)(x) = b(h^{-1}x)$ for $b \in B^{X,A}$, $h \in H$ and $x \in X$. It is easy to see that this action is continuous. The *(semi-restricted) wreath product* $B \wr^A_X H$ is now defined as

$$B \wr^A_X H = B^{X,A} \rtimes_\alpha H \tag{4.5}$$

equipped with the product topology. By [Cor17, Proposition 2.4] it is a lcsc group. Theorem D is now a consequence of the following theorem.

Theorem 4.7. *Let A, B, X and H be as above. Suppose that B is non-compact and $|X| \geqslant 2$. Then, $B \wr_X^A H$ belongs to class $\mathcal{S}$ if and only if B is amenable, the stabilizer $\mathrm{Stab}_H(x)$ of every point $x \in X$ is amenable and H belongs to class $\mathcal{S}$.*

Proof. If $B \wr_X^A H$ belongs to class $\mathcal{S}$, then the subgroups H and $B \times B$ do. Hence, B must be amenable. For every point $x_0 \in X$ the subgroup $B \times \mathrm{Stab}_H(x_0)$ belongs to class $\mathcal{S}$ and since B is non-compact, this implies the amenability of $\mathrm{Stab}_H(x_0)$.

Conversely, suppose that H belongs to class $\mathcal{S}$ and that B and all stabilizers $\mathrm{Stab}_H(x)$ are amenable. Denote by $X = \bigcup_{i \in I} X_i$ the partition of X into the orbits of $H \curvearrowright X$ and fix $x_i \in X_i$ for all $i \in I$. Write $B_i = B^{X_i, A}$ and $H_i = \mathrm{Stab}_H(x_i)$.

Step 1. Step 1. Each $B \wr_{X_i}^A H$ belongs to class $\mathcal{S}$. Fix $i \in I$. To prove this step, we proceed along the lines of [BO08, Corollary 15.3.6]. By Lemma 4.3 and Corollary 4.6, it suffices to prove the existence of a continuous map $\zeta_i : B_i \to M(H/H_i)^+ \cong \ell_1(X_i)^+$ satisfying

$$\lim_{b \to \infty} \frac{\left\| h \cdot \zeta_i(b) - \zeta_i\big(\alpha_h(b)\big) \right\|_1}{\|\zeta_i(b)\|_1} = 0 \qquad \text{and} \qquad \lim_{b \to \infty} \frac{\|\zeta_i(aba') - \zeta_i(b)\|_1}{\|\zeta_i(b)\|_1} = 0 \tag{4.6}$$

uniformly on compact sets for $h \in H$ and $a, a' \in B_i$.

By [Str74] every lcsc group G admits a continuous proper length function, i.e. a continuous proper function $\ell : G \to \mathbb{R}^+$ satisfying $\ell(gh) \leqslant \ell(g) + \ell(h)$ and $\ell(g) = \ell(g^{-1})$ for all $g, h \in G$. Fix such continuous, proper length functions $\ell_B : B \to \mathbb{R}^+$ and $\ell_H : H \to \mathbb{R}^+$. Define the function

$$f : X_i \to \mathbb{R}^+ : x \mapsto \inf_{\substack{h \in H \\ h x_i = x}} \ell_H(h).$$

Note that f is proper and that $f(hx) \leqslant \ell_H(h) + f(x)$ for $x \in X$ and $h \in H$. Define

$$g : B \to \mathbb{R}^+ : b \mapsto \inf_{a, a' \in A} \ell_B(aba'),$$

and note that $g(bb') \leqslant g(b) + g(b') + N$, where $N = \sup_{a \in A} \ell_B(a)$.

Define $\zeta_i : B_i \to \ell^1(X_i)^+$ by

$$\zeta_i(b)(x) = \begin{cases} g\big(b(x)\big) + f(x) & \text{if } x \in \mathrm{supp}_A(b), \\ 0 & \text{otherwise} \end{cases}$$

for $b \in B_i$ and $x \in X_i$.

We prove that ζ_i satisfies (4.6). Fix $h \in H$ and $a, a', b \in B_i$. Denote $b' = aba'$, $S = \mathrm{supp}_A b$, $S' = \mathrm{supp}_A b'$ and $T = \mathrm{supp}_A a \cup \mathrm{supp}_A a'$. We have

$$\left\| h \cdot \zeta(b) - \zeta\big(\alpha_h(b)\big) \right\|_1 = \sum_{x \in hS} |f(h^{-1}x) - f(x)| \leqslant |S| \, \ell_H(h)$$

and

$$\begin{aligned} \|\zeta(b') - \zeta(b)\|_1 &= \sum_{x \in T} \left| g\big(b'(x)\big) - g\big(b(x)\big) \right| + \sum_{x \in T \cap (S \triangle S')} f(x) \\ &\leqslant \sum_{x \in T} \Big(g\big(a'(x)\big) + g\big(a(x)\big) + 2N \Big) + \sum_{x \in T \cap (S \triangle S')} f(x) \\ &\leqslant \|\zeta(a)\|_1 + \|\zeta(a')\|_1 + 2N \, |T|. \end{aligned}$$

So, it suffices to prove that

$$\lim_{b \to \infty} \|\zeta_i(b)\|_1 = +\infty \qquad \text{and} \qquad \lim_{b \to \infty} \frac{|\mathrm{supp}_A b|}{\|\zeta_i(b)\|_1} = 0.$$

To prove the first, suppose that $\|\zeta(b)\|_1 \leqslant M$ for some $M > 0$. Then, $f(x) \leqslant M$ and $g\big(b(x)\big) \leqslant M$ for every $x \in \operatorname{supp}_A(b)$. Hence, we have $b \in C = \prod_{x \in X_i} C_x$, where $C_x = \{b \in B \mid g(b) \leqslant M\}$ for $x \in F = \{x \in X \mid f(x) \leqslant M\}$ and $C_x = A$ otherwise. Since F is finite and each C_x is compact, it follows that C is compact.

Finally, to prove that $|\operatorname{supp}_A b|/\|\zeta_i(b)\|_1 \to 0$ if $b \to \infty$, take $b \in B$ such that $|\operatorname{supp}_A b|/\|\zeta_i(b)\|_1 \geqslant \delta$ for some $\delta > 0$. Denote $D = \{x \in X_i \mid f(x) \leqslant 2/\delta\}$. Then,

$$\frac{2}{\delta}\big(|\operatorname{supp}_A b| - |D|\big) \leqslant \frac{2}{\delta}|\operatorname{supp}_A b \setminus D| \leqslant \|\zeta_i(b)\|_1 \leqslant \frac{1}{\delta}|\operatorname{supp}_A b|$$

and thus $|\operatorname{supp}_A b| \leqslant 2|D|$. We get $\|\zeta_i(b)\|_1 \leqslant \frac{2}{\delta}|D|$. But, by the previous, the set $\{b \in B \mid \|\zeta(b)\|_1 \leqslant 2|D|/\delta\}$ is compact and so is $\{b \in B \mid |\operatorname{supp}_A b|/\|\zeta_i(b)\|_1 \geqslant \delta\}$.

Step 2. Step 2. Construction of maps $\xi_i : B_i \to \operatorname{Prob}(H)$ satisfying (4.7) below. Fix $i \in I$, $\varepsilon > 0$ and a compact $K \subseteq H$. In this step, we construct a Borel map $\xi_i : B_i \to \operatorname{Prob}(H)$ such that

$$\big\|\xi_i\big(\alpha_h(b)\big) - h \cdot \xi_i(b)\big\| \leqslant \varepsilon \qquad \text{and} \qquad \xi_i(aba') = \xi_i(b) \tag{4.7}$$

for all $b \in B_i \setminus A^{X_i}$, all $h \in K$ and all $a, a' \in A^{X_i}$. Note that the difference with the previous step is that we want the map ξ_i to satisfy (4.7) for all $b \in B_i \setminus A^{X_i}$, instead of $b \in B_i \setminus L$ for L some (possibly large) compact set.

Since H_i is amenable, the action $H \curvearrowright H/H_i$ by left translation is amenable. Identifying $X_i \cong H/H_i$ and using Proposition 2.5, we find a map $\mu : X_i \to \operatorname{Prob}(H)$ such that $\|h \cdot \mu(x) - \mu(hx)\| < \varepsilon$ for every $h \in K$ and every $x \in X_i$. Now, define $\xi_i : B_i \to \operatorname{Prob}(H)$ by

$$\xi_i(b) = \frac{1}{|\operatorname{supp}_A b|} \sum_{x \in \operatorname{supp}_A b} \mu(x)$$

for $b \in B_i \setminus A^{X_i}$. For $b \in B_i$, set $\xi_i(b) = \delta_e$. One easily checks that ξ_i satisfies (4.7).

Step 3. Step 3. $B \wr_X^A H$ is bi-exact. Fix $\varepsilon > 0$ and take compact sets $C \subseteq B^{X,A}$ and $K \subseteq H$. By Lemma 3.2 and Corollary 4.6, it suffices to prove that there exists a compact set $D \subseteq B^{X,A}$ and a Borel map $\zeta : B^{X,A} \to \operatorname{Prob}(H)$ such that

$$\big\|h \cdot \zeta(b) - \zeta\big(\alpha_h(b)\big)\big\| \leqslant \varepsilon \qquad \text{and} \qquad \|\zeta(aba') - \zeta(b)\| \leqslant \varepsilon \tag{4.8}$$

for all $h \in K$, $a, a' \in C$ and $b \in B^{X,A} \setminus D$.

Take compact sets $C_i \subseteq B_i$ and a finite subset $I_0 \subseteq I$ such that $C \subseteq \prod_{i \in I} C_i$ and such that $C_i = A^{X_i}$ for all $i \in I \setminus I_0$. For $i \in I_0$, the fact that $B \wr_{X_i}^A H$ belongs to class $\mathcal{S}$, allows us to take a compact set $D_i \subseteq B_i$ and a Borel map $\zeta_i : B_i \to \operatorname{Prob}(H)$ such that $\big\|h \cdot \zeta_i(b) - \zeta_i\big(\alpha_h(b)\big)\big\| \leqslant \varepsilon$ and $\|\zeta_i(aba') - \zeta_i(b)\| \leqslant \varepsilon$ for $h \in K$, $a, a' \in C_i$ and $b \in B_i \setminus D_i$. By enlarging D_i, we can assume that $A^{X_i} \subseteq D_i$ and $C_i^{-1} A^{X_i} C_i^{-1} \subseteq D_i$. For $i \in I \setminus I_0$, we take for $\zeta_i : B_i \to \operatorname{Prob}(H)$ the map ξ_i from step 2 and set $D_i = A^{X_i}$.

For $b \in B^{X,A}$ and $i \in I$, we denote by $b_i \in B^{X_i,A}$ the restriction of b to X_i. We also denote $I_b = \big\{i \in I \bigm| b_i \notin A^{X_i}\big\}$. Define $\zeta : B^{X,A} \to \operatorname{Prob}(H)$ by

$$\zeta_i(b) = \frac{1}{|I_b|} \sum_{i \in I_b} \zeta_i(b_i)$$

for $b \in B^{X,A} \setminus A^X$ and $\zeta_i(b) = \delta_e$ for $b \in A^X$. One easily checks that (4.8) holds for $D = \prod_{i \in I} D_i$, since $I_b = I_{aba'}$ for $b \in B^{X,A} \setminus D$ and $a, a' \in C$. □

5 Class $\mathcal{S}$ is closed under measure equivalence

In this section, we prove Theorem E. As mentioned in the introduction, exactness is preserved under measure equivalence. So, it suffices to prove that property (S) (i.e. the existence of a map $\eta : G \to \operatorname{Prob}(G)$ satisfying

(1.1)) is a measure equivalence invariant. In order to prove that, we will use the characterization of measure equivalence in terms of cross section equivalence relations from [KKR18, Theorem A] and introduce a notion of property (S) for these relations.

Recall that a countable, Borel equivalence relation $\mathcal{R}$ on a standard probability space (X, μ) is an equivalence relation on X such that $\mathcal{R} \subseteq X \times X$ is a Borel subset and such that all orbits are countable. We say that $\mathcal{R}$ is non-singular for the measure μ if $\mu(E) = 0$ implies that $\mu([E]_{\mathcal{R}}) = 0$ for all measurable $E \subseteq X$. Here, $[E]_{\mathcal{R}} = \{x \in X \mid \exists y \in E : x \sim_{\mathcal{R}} y\}$. We say that $\mathcal{R}$ is *ergodic* if $E = [E]_{\mathcal{R}}$ implies that $\mu(E) = 0$ or $\mu(E) = 1$. We denote $\mathcal{R}^{(2)} = \{(x, y, z) \mid x \sim_{\mathcal{R}} y \sim_{\mathcal{R}} z\}$.

A Borel subset $\mathcal{W} \subseteq \mathcal{R}$ is called *bounded* if the number of elements in its sections is bounded, i.e. if there exists a $C > 0$ such that

$$|{}_x\mathcal{W}| = |\{y \in X \mid (x, y) \in \mathcal{W}\}| < C \quad \text{and} \quad |\mathcal{W}_y| = |\{x \in X \mid (x, y) \in \mathcal{W}\}| < C$$

for a.e. $x, y \in X$. We say that $\mathcal{W}$ is *locally bounded* if for every $\varepsilon > 0$, there exists a Borel subset $E \subseteq X$ with $\mu(X \setminus E) \leqslant \varepsilon$ such that $\mathcal{W} \cap (E \times E)$ is bounded.

The *full group* $[\mathcal{R}]$ is the group of all Borel automorphisms $\varphi : X \to X$, identified up to almost everywhere equality, such that $\operatorname{graph} \varphi = \{(\varphi(x), x)\}_{x \in X}$ is contained in $\mathcal{R}$. The *full pseudo group* $[[\mathcal{R}]]$ is the set of all partial Borel isomorphisms $\varphi : A \to B$ for Borel sets $A, B \subseteq X$ whose graph is contained in $\mathcal{R}$. Again, these partial isomorphisms are identified up to almost everywhere equality. Every bounded Borel subset $\mathcal{W} \subseteq \mathcal{R}$ can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$. For more information about countable equivalence relations, see for instance [FM77].

Let G be a lcsc group and $G \curvearrowright (X, \mu)$ a probability measure preserving (pmp) action. We say that the action $G \curvearrowright (X, \mu)$ is *essentially free* if the set

$$\{x \in X \mid \exists g \in G : gx = x\}$$

is a null set. Note that this set is Borel by [MRV13, Lemma 10].

The notion of a cross section equivalence relation was originally introduced by Forrest in [For74]. A more recent, self-contained treatment for unimodular groups can be found in [KPV15]. Given an essentially free pmp action $G \curvearrowright (X, \mu)$ on a standard probability space, a *cross section* is a Borel subset $X_1 \subseteq X$ with the following two properties.

(i) There exists a neighbourhood $\mathcal{U} \subseteq G$ of identity such that the action map $\mathcal{U} \times X_1 \to X : (g, x) \mapsto gx$ is injective.

(ii) The subset $G \cdot X_1 \subseteq X$ is co-null.

By [For74, Theorem 4.2] such a cross section always exists. Note that the first condition implies that the action map $\theta : G \times X_1 \to X : (g, x) \mapsto gx$ is countable-to-one and hence maps Borel sets to Borel sets. In particular, the set $G \cdot X_1$ in the second condition is Borel.

By removing a G-invariant null set from X, we can always assume that $G \cdot X_1 = X$ and that $G \curvearrowright X$ is really free. Using [Kec95, 18.10 and 18.14], we can take Borel maps $\pi : X \to X_1$ and $\gamma : X \to G$ such that $x = \gamma(x) \cdot \pi(x)$ for all $x \in X$. Similarly, denoting by $\mathcal{R}_G$ the image of the map $G \times X \to X \times X : (g, x) \mapsto (gx, x)$, we can take a Borel map $\omega : \mathcal{R}_G \to G$ satisfying $\omega(x, y)y = x$ for $y \in G \cdot x$. Moreover, ω is a 1-cocycle in the sense that $\omega(x, y)\omega(y, z) = \omega(x, z)$ for all $y, z \in G \cdot x$.

The *cross section equivalence relation* associated to X_1 is defined by

$$\mathcal{R} = \mathcal{R}_G \cap (X_1 \times X_1) = \{(x, y) \in X_1 \times X_1 \mid y \in G \cdot X_1\}.$$

The measurable space X_1 admits a unique probability measure μ_1 and a unique number $0 < \operatorname{covol}(X_1) < +\infty$ such that

$$(\lambda_G \otimes \mu_1)(\mathcal{W}) = \operatorname{covol}(X_1) \int_X |\mathcal{W} \cap \theta^{-1}(x)| \, \mathrm{d}\mu(x) \tag{5.1}$$

for all measurable $\mathcal{W} \subseteq G \times X_1$. The relation $\mathcal{R}$ is a non-singular, countable, Borel equivalence relation for this probability measure μ_1.

We will use the following easy lemma throughout the rest of this section.

Lemma 5.1. *Let G be a lcsc group and $G \curvearrowright (X,\mu)$ an essentially free, pmp action. Let $X_1 \subseteq X$ be a cross section and $\mathcal{R}$ the associated cross section equivalence relation.*

(a) *If $K \subseteq G$ is compact, then the set $\mathcal{W} = \{(x,y) \in \mathcal{R} \mid \omega(x,y) \in K\}$ is a bounded subset of $\mathcal{R}$.*

(b) *If $\mathcal{W} \subseteq \mathcal{R}$ is a locally bounded set and $\varepsilon > 0$, then there exists a Borel subset $E \subseteq X_1$ with $\nu(E) < \varepsilon$ such that $\omega\big(\mathcal{W} \cap (E \times E)\big)$ is relatively compact.*

Proof. Statement (a) follows easily from the fact that there is a neighbourhood of the unit $e \in G$ for which the map $\mathcal{U} \times X_1 \to X : (g,x) \mapsto gx$ is injective.

Since every bounded Borel subset can be written as a finite union of graphs of elements in $[[\mathcal{R}]]$, it suffices to prove (b) for graph(φ) with $\varphi \in [[\mathcal{R}]]$, but this can be done easily by taking $E = \big\{x \in X \bigm| \omega\big(\alpha(x),x\big) \in K\big\}$ for K a compact set that is large enough. □

We define property (S) on the level of non-singular, countable, Borel equivalence relations as follows.

Definition 5.2. Let $\mathcal{R}$ be a non-singular, countable, Borel equivalence relation on a standard measure space (X,μ). We say that $\mathcal{R}$ has *property (S)* if there exists a Borel map η assigning to all $(x,y) \in \mathcal{R}$ a probability measure on the orbit of y such that for all $\varepsilon > 0$ and $\varphi,\psi \in [\mathcal{R}]$, the set

$$\big\{(x,y) \in \mathcal{R} \bigm| \big\|\eta\big(\varphi(x),\psi(y)\big) - \eta(x,y)\big\|_1 \geqslant \varepsilon\big\} \tag{5.2}$$

is locally bounded.

Remark 5.3. To be entirely rigorous, we can view η as a Borel map $\mathcal{R}^{(2)} \to [0,1]$ such that $\sum_{\substack{z\in X\\ z\sim x}} \eta(x,y,z) = 1$ for a.e. $(x,y) \in \mathcal{R}$.

We prove that the above notion of property (S) is compatible with taking cross section equivalence relations.

Proposition 5.4. *Let G be a lcsc group and $G \curvearrowright (X,\mu)$ an essentially free, ergodic, pmp action. Let $X_1 \subseteq X$ be a cross section and $\mathcal{R}$ the associated cross section equivalence relation. Then, G has property (S) if and only if $\mathcal{R}$ has property (S).*

Proof. As before, we fix Borel maps $\gamma : X \to G$ and $\pi : X \to X_1$ such that $x = \gamma(x) \cdot \pi(x)$ for a.e. $x \in X$. First, assume that G has property (S). Let $\eta : G \to \text{Prob}(G)$ be a map satisfying (1.1). Define for each $x \in X$ a map

$$\pi_x : G \to X_1 : g \mapsto \pi(g^{-1}x).$$

Note that π_x is a Borel map from G to the $\mathcal{R}$-orbit of $\pi(x)$. We define the map η' as in Definition 5.2 by

$$\eta'(x,y) = (\pi_x)_*\eta\big(\omega(x,y)\big)$$

for $(x,y) \in \mathcal{R}$. Note that indeed every $\eta'(x,y)$ is a probability measure on the $\mathcal{R}$-orbit of x. To prove that η' satisfies (5.2), fix $\varepsilon,\delta > 0$ and $\varphi,\psi \in [\mathcal{R}]$. It suffices to find a Borel set $E \subset X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that the set

$$\big\{(x,y) \in \mathcal{R} \cap (E \times E) \bigm| \big\|\eta'\big(\varphi(x),\psi(y)\big) - \eta'(x,y)\big\|_1 \geqslant \varepsilon\big\} \tag{5.3}$$

is bounded.

By Lemma 5.1, we find a compact set $K \subseteq G$ and a measurable $E \subseteq X_1$ with $\mu_1(X_1 \setminus E) < \delta$ such that $\omega\big(\varphi(x),x\big) \in K$ and $\omega\big(y,\psi(y)\big) \in K$ for all $x,y \in E$. Take a compact set $L \subseteq G$ such that $\|\eta(gkh) - g\cdot\eta(k)\|_1 < \varepsilon$ for all $g,h \in K$ and all $k \in G \setminus L$. We claim that

$$\big\|\eta'\big(\varphi(x),\psi(y)\big) - \eta'(x,y)\big\|_1 < \varepsilon \tag{5.4}$$

whenever $(x,y) \in \mathcal{R} \cap (E \times E)$ and $\omega(x,y) \in G \setminus L$. Assuming the claim is true, the set (5.3) is contained in the set of all $(x,y) \in \mathcal{R}$ with $\omega(x,y) \in L$ which is bounded by Lemma 5.1. To prove (5.4), fix $(x,y) \in \mathcal{R} \cap (E \times E)$ with $\omega(x,y) \in G \setminus L$. We have

$$\begin{aligned}\big\|\eta'\big(\varphi(x),\varphi(y)\big) - \eta'(x,y)\big\|_1 &= \big\|(\pi_{\varphi(x)})_*\eta\big(\omega(\varphi(x),\psi(y))\big) - (\pi_x)_*\eta\big(\omega(x,y)\big)\big\|_1\\ &= \big\|\eta\big(\omega(\varphi(x),\psi(y))\big) - \omega\big(\varphi(x),x\big)\cdot\eta\big(\omega(x,y)\big)\big\|_1 < \varepsilon,\end{aligned}$$

where we used that $\pi_x(g) = \pi_{\varphi(x)}\big(\omega(\varphi(x),x)g\big)$ and the cocycle identity for ω. Hence, (5.4) is proved.

Conversely, assume that $\mathcal{R}$ has property (S) and let η be a map as in the definition. Choose an arbitrary $\xi \in \operatorname{Prob}(G)$ and define

$$\eta' : G \to \operatorname{Prob}(G) : g \mapsto \int_X \Big(\sum_{\substack{z\in X_1 \\ z\sim\pi(x)}} \eta\big(\pi(gx),\pi(x),z\big)\omega(gx,z)\cdot\xi\Big)\,\mathrm{d}\mu(x).$$

We prove that η' satisfies (1.1). Fix a compact, symmetric neighbourhood K of the unit e in G and an $\varepsilon > 0$. Take a compact, symmetric subset $L \subseteq G$ such that $F = \gamma^{-1}(L)$ satisfies $\mu(F) \geqslant 1-\varepsilon$. Denote $\kappa = \lambda_G(L)/\operatorname{covol}(X_1)$. By Lemma 5.1, the set $\mathcal{W} = \{(x,y) \in \mathcal{R} \mid \omega(x,y) \in LKL\}$ is bounded Borel. Writing $\mathcal{W}$ as a union of finitely many elements of $[[\mathcal{R}]]$ and using (5.2), we see that the set

$$\mathcal{V} = \{(x,y) \in \mathcal{R} \mid \exists (x,x'),(y,y') \in \mathcal{W}, \|\eta(x',y') - \eta(x,y)\|_1 \geqslant \varepsilon\}$$

is locally bounded. Denoting $\delta = \varepsilon/\kappa$ and using Lemma 5.1, we find a compact set $C \subseteq G$ and a measurable $E \subseteq X_1$ with $\mu_1(E) \geqslant 1-\delta$ such that $\omega\big(\mathcal{V}\cap(E\times E)\big) \subseteq C$. We conclude that $\|\eta(x',y')-\eta(x,y)\|_1 < \varepsilon$ whenever $(x,y) \in \mathcal{R}\cap(E\times E)$ with $(x,x')\in\mathcal{W}$, $(y,y')\in\mathcal{W}$ and $\omega(x,y) \in G\setminus C$.

Denote $D = LCL$. We conclude the proof by proving that

$$\|\eta'(gkh) - g\cdot\eta'(k)\| < 4\kappa\delta + 9\varepsilon = 13\varepsilon \tag{5.5}$$

for all $g,h\in K$ and $k \in G\setminus D$. So, fix $g,h \in K$ and $k \in G\setminus D$. Applying the change of variables $x \mapsto h^{-1}x$ and using that $\omega(gkx,z) = g\omega(kx,z)$, a straightforward calculation yields

$$\|\eta'(gkh) - g\cdot\eta'(k)\| \leqslant \int_X \big\|\eta\big(\pi(gkx),\pi(h^{-1}x)\big) - \eta\big(\pi(kx),\pi(x)\big)\big\|_1\,\mathrm{d}\mu(x).$$

Since $g,h^{-1}\in K$, we have that $\big(\pi(gkx),\pi(kx)\big)\in\mathcal{W}$ and $\big(\pi(h^{-1}x),\pi(x)\big)\in\mathcal{W}$ whenever $x\in X$ is such that $gkx, h^{-1}x, kx, x \in F = \gamma^{-1}(L)$. Moreover, for such an x we also have $\omega\big(\pi(kx),\pi(x)\big) \in LkL \subseteq G\setminus C$. Hence,

$$\big\|\eta\big(\pi(gkx),\pi(h^{-1}x)\big) - \eta\big(\pi(kx),\pi(x)\big)\big\|_1 < \varepsilon \tag{5.6}$$

whenever $gkx, h^{-1}x, kx, x \in F$, $\pi(x) \in E$ and $\pi(kx)\in E$.

Since $\mu(F) \geqslant 1-\varepsilon$, we can find a measurable set F' with $\mu(F') \geqslant 1-4\varepsilon$ such that $gkx, h^{-1}x, kx, x \in F$ for every $x \in F'$. Moreover, the map $\theta : G\times X_1 \to X$ is injective on the image A of the map $x\mapsto\big(\gamma(x),\pi(x)\big)$. Hence by (5.1), we have that $\operatorname{covol}(X_1)\,\mu\big(\theta(\mathcal{U})\big) = (\lambda_G\otimes\mu_1)(\mathcal{U})$ for all $\mathcal{U}\subseteq A$. It follows that for measurable $S\subseteq X_1$, we have that

$$\mu\big(\pi^{-1}(S)\cap F\big) = \operatorname{covol}(X_1)^{-1}(\lambda_G\times\mu_1)\big(A\cap(L\times S)\big) \leqslant \frac{\lambda_G(L)}{\operatorname{covol}(X_1)}\mu_1(S) = \kappa\mu_1(S).$$

Applying this to $\pi^{-1}(X_1\setminus E)\cap F$ and using the definition F' above, we conclude that (5.6) holds on a set whose complement has at most measure $4\varepsilon + 2\kappa\delta$ and hence that (5.5) holds. ☐

The proof of Theorem E is now easy.

Proof of Theorem E. As mentioned in the introduction exactness is a measure equivalence invariant by [DL15, Corollary 2.9] and [DL14, Theorem 0.1 (6)].

The characterization of measure equivalence in terms of stable isomorphism of cross section equivalence relations (see [KKR18, Theorem A] and [KKR17, Theorem A]) together with Proposition 5.4 yields that property (S) is preserved under measure equivalence. ☐

6 Class $\mathcal{S}$ and unique prime factorization

In [HI17], Houdayer and Isono introduce the following property.

Definition 6.1. Let $(M, \mathcal{H}, J, \mathfrak{P})$ be a von Neumann algebra in standard form. We say that M satisfies the *strong condition (AO)* if there exist C*-algebras $A \subseteq M$ and $\mathcal{C} \subseteq B(\mathcal{H})$ such that

- A is exact and σ-weakly dense in M,
- $\mathcal{C}$ is nuclear and contains A,
- all commutators $[c, JaJ]$ for $c \in \mathcal{C}$ and $a \in A$ belong to the compact operators $K(\mathcal{H})$.

Note that the definition in [HI17, Definition 2.6] requires A and $\mathcal{C}$ to be unital. However, by [BO08, Proposition 2.2.1 and Proposition 2.2.4] this is not essential.

In [HI17, Theorems B], Houdayer and Isono provide a unique prime factorization theorem for non-amenable factors satisfying strong condition (AO). A slightly more general version, removing the condition that the unknown tensor product factors N_i have a state with large centralizers, was later proved by Ando, Haagerup, Houdayer, and Marrakchi in [AHHM18, Application 4]. Theorem F now follows immediately by combining these theorems with the following result.

Proposition 6.2. *Let G be a lcsc group in class $\mathcal{S}$, then its group von Neumann algebra $L(G)$ satisfies strong condition (AO).*

Proof. Recall that $L(G)$ is in standard form on $L^2(G)$, where the anti-unitary operator J is given by $(J\xi)(t) = \delta_G(t)^{-1/2}\overline{\xi(t^{-1})}$. Here, where δ_G denotes the modular function of G. Straightforward calculation yields

$$\big(J\lambda(f)J\xi\big)(s) = \int_G \overline{f(t)}\delta_G(t)^{1/2}\xi(st)\,\mathrm{d}t.$$

Let $A = \mathrm{C}^*_r(G)$ be the reduced group C*-algebra of G. Then, obviously A is exact and σ-weakly dense in $L(G)$. By Theorem B and [Ana02, Theorem 5.3], the algebra $C(h^uG) \rtimes G$ is nuclear. Now, the inclusion $C(h^uG) \subseteq C^u_b(G) \hookrightarrow B(L^2(G))$ together with the unitary representation $g \mapsto \lambda_g$ induces a *-morphism $\pi : C(h^uG) \rtimes G \to B(L^2(G))$. Let $\mathcal{C}$ be the image of this *-morphism. The algebra $\mathcal{C}$ is nuclear as a quotient of a nuclear C*-algebra, and obviously contains A. Note that $C_c\big(G, C(h^uG)\big)$ is a dense subalgebra in $C(h^uG) \rtimes G$. Identifying an element $h \in C_c\big(G, C(h^uG)\big) \subseteq C(h^uG) \rtimes G$ with a function on $G \times G$ that is compactly supported in the first component, we get that the action $\pi(h)$ on a $\xi \in L^2(G)$ is given by

$$\big(\pi(h)\xi\big)(s) = \int_G h(t, s)\xi(t^{-1}s)\,\mathrm{d}t.$$

Denote by $\mathcal{C}_0$ the image of $C_c\big(G, C(h^uG)\big)$ under π.

For $f \in C_c(G)$ and $\pi(h) \in \mathcal{C}_0$, we prove that $T = [\pi(h), J\lambda(f)J] \in K(L^2(G))$. A straightforward calculation yields that for $\xi \in L^2(G)$ and $s \in G$, we have

$$\begin{aligned}(T\xi)(s) &= \int_G\int_G \big(h(t,s) - h(t,su)\big)\overline{f(u)}\delta_G(u)^{1/2}\xi(t^{-1}su)\,\mathrm{d}t\,\mathrm{d}u\\ &= \int_G k(s,u)\xi(u)\,\mathrm{d}u,\end{aligned}$$

where

$$k(s,u) = \int_G \big(h(t,s) - h(t,tu)\big)\overline{f(s^{-1}tu)}\delta_G(s^{-1}tu)^{1/2}\,\mathrm{d}t.$$

Let $(K_n)_n$ be an increasing sequence of compact subsets of G such that $G = \bigcup_n K_n$. Set $k_n(s,u) = \chi_{K_n}(s)k(s,u)$ and define the operators $T_n \in B(L^2(G))$ by

$$(T_n\xi)(s) = \int_G k_n(s,u)\xi(u)\,\mathrm{d}u.$$

Note that since $f \in C_c(G)$ and h is compactly supported in the first component, we have that each $k_n \in L^2(G \times G)$ and hence that T_n is compact. Moreover, $T_n \to T$ in norm. Indeed, if $L \subseteq G$ is a compact subset containing the support of f and of (the first component of) h, then

$$\begin{aligned}\|T\xi - T_n\xi\|^2 &\leqslant \sup_{s \in G \setminus K_n} \sup_{t,u \in L} |h(t,s) - h(t,su)|^2 \, \mu(L)^2 \, \|J\lambda(|f|)J|\xi|\|_2^2 \\ &= \sup_{s \in G \setminus K_n} \sup_{t,u \in L} |h(t,s) - h(t,su)|^2 \, \mu(L)^2 \, \|f\|_1^2 \, \|\xi\|_2^2\end{aligned}$$

and $\limsup_{s \to \infty} |h(t,s) - h(t,su)|^2 = 0$ uniformly on compact sets for $t, u \in G$. We conclude that T itself is compact. □

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