# Rigidity for von Neumann algebras given by locally compact groups

Tobe Deprez[1]
Joint work with A. Brothier, S. Vaes

KU Leuven

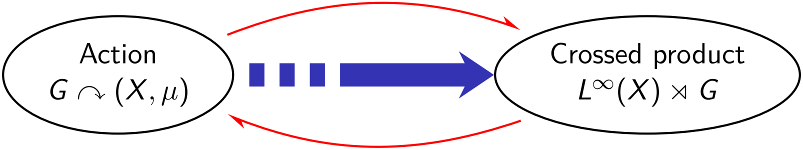
YMC*A 2017, Copenhagen

[1] Supported by a PhD fellowship of the Research Foundation Flanders (FWO)

# Contents

# Contents

# Introduction




- **Crossed product**: $M = L^\infty(X) \rtimes G$ generated by $\{u_g\}_{g\in G}$ and $L^\infty(X,\mu)$ such that $u_g f u_g^* = \sigma_g(f)$ for $g \in G$ and $f \in L^\infty(X,\mu)$

## Standing assumption

- **essentially free**: $\{x \in X \mid \exists g \in G : gx = x\}$ is a null set,
- **ergodic**: if $\mu(gA\Delta A) = 0$ for all $g \in G$, then $A$ is null or co-null.
- **prob. measure preserving**: $\mu$ is prob. measure and $\mu(gA) = \mu(A)$.

⇒ $M$ is a factor

# Contents

# Countable group $\Gamma$

- All $L^\infty(X) \rtimes \Gamma$ are isomorphic for $\Gamma$ amenable.
- $L^\infty(X) \rtimes \Gamma$ only depends on **orbit equivalence relation**

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(gx, x) \mid x \in X, g \in \Gamma\}$$

Three "levels" of isomorphisms $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$

conjugate $\Gamma \curvearrowright (X, \mu) \cong \Lambda \curvearrowright (Y, \nu)$ $\Longrightarrow$ orbit equivalent $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$ $\Longrightarrow$ W*-equivalent $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$

**Theorem (Singer, 1955)**

*If there exists an isomorphism*

$$\Psi : L^\infty(X) \rtimes \Gamma \xrightarrow{\sim} L^\infty(Y) \rtimes \Lambda \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

*then* $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

# Cartan subalgebras

**Theorem (Singer, 1955)**

*If there exists an isomorphism*

$$\Psi : L^\infty(X) \rtimes \Gamma \xrightarrow{\sim} L^\infty(Y) \rtimes \Lambda \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

*then* $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

- $L^\infty(X)$ is a **Cartan subalgebra**

**Definition**

$A \subseteq M$ is a **Cartan subalgebra** if

(i) $A$ is maximal abelian (i.e. $A' \cap M = A$),

(ii) $\mathcal{N}_M(A) = \{u \in M \mid u \text{ unitary}, uAu^* = A\}$ generates $M$,

(iii) $\exists E : M \to A$ conditional expectation.

# Cartan subalgebras

**Theorem (Singer, 1955)**

*If there exists an isomorphism*

$$\Psi : L^\infty(X) \rtimes \Gamma \xrightarrow{\sim} L^\infty(Y) \rtimes \Lambda \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

*then* $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

- $L^\infty(X)$ is a **Cartan subalgebra**

  ➡ if $L^\infty(X)$ has unique Cartan subalgebra (up to conjugacy)

$$\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y) \iff L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$$

# Uniqueness of Cartan subalgebras

$L^\infty(X) \rtimes \Gamma$ has unique Cartan (up to unitary conjugacy) if

- **(Ozawa-Popa, 2010)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Chifan-Sinclair, 2013)** $\Gamma$ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Popa-Vaes, 2014)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ arbitrary
- **(Popa-Vaes, 2014)** $\Gamma$ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ arbitrary

**Theorem (Gaboriau, 2000)**

*$\mathcal{R}(\mathbb{F}_n \curvearrowright X) \ncong \mathcal{R}(\mathbb{F}_m \curvearrowright X)$ if $n \neq m$.*

**Corollary**

*$L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$.*

# Uniqueness of Cartan subalgebras

$L^\infty(X) \rtimes \Gamma$ has unique Cartan (up to unitary conjugacy) if

- **(Ozawa-Popa, 2010)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X,\mu)$ profinite
- **(Chifan-Sinclair, 2013)** $\Gamma$ hyperbolic and $\Gamma \curvearrowright (X,\mu)$ profinite
- **(Popa-Vaes, 2014)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X,\mu)$ arbitrary
- **(Popa-Vaes, 2014)** $\Gamma$ hyperbolic and $\Gamma \curvearrowright (X,\mu)$ arbitrary

Actually holds for $\Gamma$ non-amenable, weakly amenable and in **Ozawa's class $\mathcal{S}$**

### Definition

$\Gamma$ belongs to Ozawa's class $\mathcal{S}$ if $\Gamma$ is exact and $\exists \eta : \Gamma \to \text{Prob}(\Gamma)$ such that

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0$$

# Contents

# Locally compact group $G$

- $G$ lcsc and unimodular
- If $G$ is locally compact, then $L^\infty(X)$ is not Cartan in $M = L^\infty(X) \rtimes G$
  **BUT:** $\exists$ **cross section** $Y \subseteq X$, i.e.
  (i) $\exists\, \mathcal{U} \subseteq G$ neighbourhood of $e \in G$ such that $\mathcal{U} \times Y \to X : (g, y) \mapsto gy$ is injective
  (ii) $G \cdot Y = X$ (up to null sets)

  Then, the **cross section equivalence relation**

$$\mathcal{R} := \mathcal{R}(G \curvearrowright X) \cap (Y \times Y) = \{(y, z) \in Y \times Y \mid \exists g \in G : gy = z\}$$

is countable and

$$M \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N}))$$

$\Rightarrow$ $L^\infty(Y) \mathbin{\overline{\otimes}} \ell^\infty(\mathbb{N})$ is Cartan in $M$

# Contents

# Results - uniqueness of Cartan

**Definition**

$G$ has property (S) if $\exists$ a continuous map $\eta : G \to \mathcal{S}(G)$ such that

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\|_1 = 0 \quad \text{uniformly on compact sets } g, h \in G,$$

where $\mathcal{S}(G) = \{F \in L^1(G) \mid F(g) \geqslant 0; \|F\|_1 = 1\}$.

**Theorem (Brothier-D-Vaes)**

*Let $G = G_1 \times \cdots \times G_n$ with $G_i$ non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.*

# Results - uniqueness of Cartan

**Theorem (Brothier-D-Vaes)**

*Let $G = G_1 \times \cdots \times G_n$ with $G_i$ non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.*

Examples

Direct products of

- finite centre connected simple Lie groups of rank 1

  **e.g.** $\mathrm{SO}(n, 1)$; $\mathrm{SU}(n, 1)$; $\mathrm{Sp}(n, 1)$
- automorphism groups of trees and hyperbolic graphs.

# Results - W*-rigidity

**Theorem (Brothier-D-Vaes)**

*Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be without compact normal subgroups. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free and irreducible. Suppose that $G_i$ non-amenable and $H_i$ non-amenable, weakly amenable and with property (S).*
*If $p(L^\infty(X) \rtimes G)p \cong q(L^\infty(Y) \rtimes H)q$, then the actions are conjugate.*

**Recall:** If $G = G_1 \times G_2$, then we say $G \curvearrowright (X, \mu)$ is irreducible if $G_i \curvearrowright (X, \mu)$ is ergodic for $i = 1, 2$.

# Contents

# Dichotomy of (Popa-Vaes, 2014)

**Theorem (Popa-Vaes, 2014)**

*Let $\Gamma$ be countable, weakly amenable and in Ozawa's class $\mathcal{S}$ (e.g. $\Gamma$ hyperbolic). Suppose $\Gamma \curvearrowright (B, \tau)$ trace-preserving. Let $M = B \rtimes \Gamma$. If $A \subseteq M$ is amenable relative to $B$, then*

- *$\mathcal{N}_M(A)''$ remains amenable relative to $B$,*
- *OR*
- *$A \preceq_M B$.*

## Proof of uniqueness of Cartan

- Suppose $\Gamma$ non-amenable, weakly amenable and in Ozawa's class $\mathcal{S}$ and $\Gamma \curvearrowright (X, \mu)$ p.m.p., free and ergodic.
- Let $B = L^\infty(X)$ and $M = B \rtimes \Gamma$. Then, $\Gamma \curvearrowright B$ trace preserving.
- For $A \subset M$ arbitrary Cartan
  - ~~$\mathcal{N}_M(A)'' = M$ remains amenable relative to $B$,~~
  - OR
  - $A \preceq_M B \implies A = uBu^*$ for some $u \in \mathcal{U}(M)$

$\square$

# A more general dichotomy

## Definition

Let $M$ be a von Neumann algebra and $G$ a group. A **co-action** is an injective, normal $*$-morphism $\Phi : M \to M \otimes L(G)$ such that

$$(\Phi \otimes 1)\Phi = (1 \otimes \Delta)\Phi,$$

where $\Delta : L(G) \to L(G) \otimes L(G)$ is the co-mult. given by $\Delta(u_g) = u_g \otimes u_g$.

## Theorem (Brothier-D-Vaes)

*Let $(M, \tau)$ be a tracial von Neumann algebra and $\Phi : M \to M \otimes L(G)$ a co-action. Suppose that $G$ is weakly amenable and with property (S). If $A \subseteq M$ is $\Phi$-amenable, then*

- *$\mathcal{N}_M(A)''$ remains $\Phi$-amenable,*
- *OR*
- *$A$ can be $\Phi$-embedded.*

# Proof of uniqueness of Cartan subalgebra for lc groups

**Theorem (Brothier-D-Vaes)**

*Let $G$ be non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.*

**Proof.**

- STP: $L(\mathcal{R})$ has unique Cartan subalgebra for $\mathcal{R}$ cross-section eq rel.
- Consider co-action $\Phi : L(\mathcal{R}) \to L(\mathcal{R}) \otimes L(G)$ given by

$$\Phi(f) = f \otimes 1, \qquad \Phi(u_\varphi) = (u_\varphi \otimes 1)V_\varphi \qquad f \in L^\infty(Y), \varphi \in [\mathcal{R}],$$

  where $V_\varphi \in L^\infty(Y) \otimes L(G)$ given by $V_\varphi(y) = \lambda_g$ when $\varphi(y) = gy$.
- For $A \subseteq M$ Cartan subalgebra

  ~~$\mathcal{N}_M(A)''$ remains $\Phi$-amenable,~~

  OR

  $A$ is $\Phi$-embedded $\implies A = uL^\infty(Y)u^*$

□

# Thank you for your attention!

# References I


- I. Chifan and T. Sinclair, On the structural theory of $\mathrm{II}_1$ factors of negatively curved groups. **Ann. Sci. Ecole Norm. S. 46** (2013), 1–33.
- D. Gaboriau, Coût des relations d equivalence et des groupes. **Invent. Math. 139** (2000), 41–98.
- C. Houdayer and S. Raum, Locally compact groups acting on trees, the type I conjecture and non-amenable von Neumann algebras. 2016. *Preprint*. arXiv: 1610.00884.
- N. Ozawa and S. Popa, On a class of $\mathrm{II}_1$ factors with at most one Cartan subalgebra. **Ann. Math. 172** (2010), 713–749.
- D. S. Ornstein and B. Weiss, Ergodic theory of amenable group actions. I: The Rohlin lemma. **Bull. Am. Math. Soc. 2** (1980), 161–165.

# References II


S. Popa and S. Vaes, Unique Cartan decomposition for $II_1$ factors arising from arbitrary actions of hyperbolic groups. **J. Reine Angew. Math. 2014** (2014), 141–198.

S. Raum, C*-simplicity of locally compact Powers groups. 2016. *Preprint*. arXiv: 1505.07793. To appear in **J. Reine Angew. Math.**

I. M. Singer, Automorphisms of Finite Factors. **Am. J. Math. 77** (1955), 117–133.