Rigidity for von Neumann algebras given by locally compact groups

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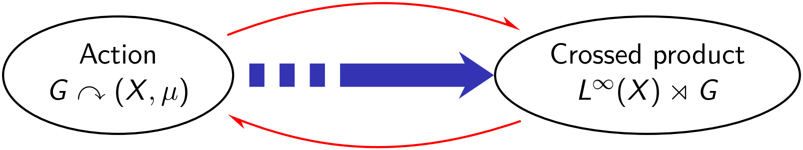
Chistmas Tree Meeting, Orsay, 2017

[1]Supported by a PhD fellowship of the Research Foundation Flanders (FWO)

Introduction

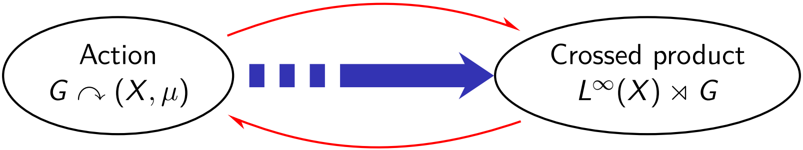


- **Crossed product**: $M = L^\infty(X) \rtimes G$ generated by $\{u_g\}_{g \in G}$ and $L^\infty(X, \mu)$ such that $u_g f u_g^* = \sigma_g(f)$ for $g \in G$ and $f \in L^\infty(X, \mu)$

Question

When is $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$?

Introduction



- **Crossed product**: $M = L^\infty(X) \rtimes G$ generated by $\{u_g\}_{g \in G}$ and $L^\infty(X, \mu)$ such that $u_g f u_g^* = \sigma_g(f)$ for $g \in G$ and $f \in L^\infty(X, \mu)$

Standing assumption

- **(essentially) free**: $\{x \in X \mid \exists g \in G : gx = x\}$ is a null set,
- **ergodic**: if $\mu(gA \Delta A) = 0$ for all $g \in G$, then A is null or co-null.
- **prob. measure preserving**: μ is prob. measure and $\mu(gA) = \mu(A)$.

$\Rightarrow$ $L^\infty(X) \times G$ is a factor

Contents

Contents

Countable group Γ

- $L^\infty(X) \rtimes \Gamma$ only depends on **orbit equivalence relation**

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(gx, x) \mid x \in X, g \in \Gamma\}$$

Three "levels" of isomorphisms $\Gamma \curvearrowright (X,\mu)$, $\Lambda \curvearrowright (Y,\nu)$

conjugate $\Gamma \curvearrowright (X,\mu) \cong \Lambda \curvearrowright (Y,\nu)$ $\Rightarrow$ orbit equivalent $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$ $\Rightarrow$ W*-equivalent $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$

- All $\mathcal{R}(\Gamma \curvearrowright X)$ (and thus $L^\infty(X) \rtimes \Gamma$) are isomorphic for Γ amenable.

Theorem (Singer, 1955)

If there exists an isomorphism

$$\Psi : L^\infty(X) \rtimes \Gamma \xrightarrow{\sim} L^\infty(Y) \rtimes \Lambda \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

then $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

Cartan subalgebras

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- $L^\infty(X)$ is a **Cartan subalgebra**

Definition

$A \subseteq M$ is a **Cartan subalgebra** if

(i) A is maximal abelian (i.e. $A' \cap M = A$),

(ii) $\mathcal{N}_M(A) = \{u \in M \mid u \text{ unitary, } uAu^* = A\}$ generates M,

(iii) $\exists E : M \to A$ conditional expectation.

Cartan subalgebras

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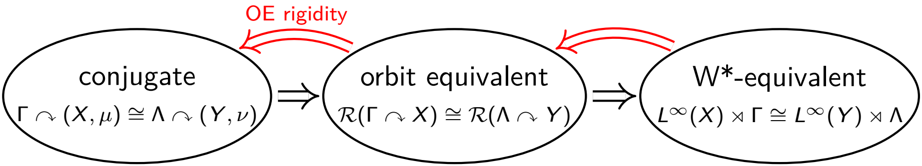
then $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

- $L^\infty(X)$ is a **Cartan subalgebra**

  ➡ if $L^\infty(X)$ has unique Cartan subalgebra (up to conjugacy)

$$\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y) \iff L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$$



Uniqueness of Cartan subalgebras

$L^\infty(X) \rtimes \Gamma$ has unique Cartan (up to unitary conjugacy) if

- **(Ozawa-Popa, 2010)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Chifan-Sinclair, 2013)** Γ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Popa-Vaes, 2014)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ arbitrary
- **(Popa-Vaes, 2014)** Γ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ arbitrary

Theorem (Gaboriau, 2000)

$\mathcal{R}(\mathbb{F}_n \curvearrowright X) \ncong \mathcal{R}(\mathbb{F}_m \curvearrowright X)$ if $n \neq m$.

Corollary

$L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$.

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Actually holds for Γ non-amenable, weakly amenable and in **Ozawa's class $\mathcal{S}$**

Definition

Γ belongs to Ozawa's class $\mathcal{S}$ if Γ is exact and $\exists \eta : \Gamma \to \text{Prob}(\Gamma)$ such that

$$\lim_{k \to \infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0$$

Contents

Locally compact group G

- G lcsc and unimodular
- If G is locally compact, then $L^\infty(X)$ is not Cartan in $M = L^\infty(X) \rtimes G$
  BUT: $\exists$ **cross section** $Y \subseteq X$, i.e.
  (i) $\exists\, \mathcal{U} \subseteq G$ neighbourhood of $e \in G$ such that $\mathcal{U} \times Y \to X : (g, y) \mapsto gy$ is injective
  (ii) $G \cdot Y = X$ (up to null sets)

  Then, the **cross section equivalence relation**

$$\mathcal{R} := \mathcal{R}(G \curvearrowright X) \cap (Y \times Y) = \{(y, z) \in Y \times Y \mid \exists g \in G : gy = z\}$$

  is countable and

$$M \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N}))$$

$\Rightarrow$ $L^\infty(Y) \mathbin{\overline{\otimes}} \ell^\infty(\mathbb{N})$ is Cartan in M

Contents

Results - uniqueness of Cartan

Definition

G has property (S) if $\exists$ a continuous map $\eta : G \to \mathcal{S}(G)$ such that

$$\lim_{k\to\infty} \|\eta(gkh) - g\cdot\eta(k)\|_1 = 0 \quad \text{uniformly on compact sets } g, h \in G,$$

where $\mathcal{S}(G) = \{F \in L^1(G) \mid F(g) \geqslant 0; \|F\|_1 = 1\}$.

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times \cdots \times G_n$ with G_i non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X,\mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.

Results - uniqueness of Cartan

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Examples

Direct products of

- finite centre connected simple Lie groups of rank 1

  e.g. $\mathrm{SO}(n,1)$; $\mathrm{SU}(n,1)$; $\mathrm{Sp}(n,1)$
- automorphism groups of trees and hyperbolic graphs.

Results - W*-rigidity

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be without compact normal subgroups. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free and irreducible. Suppose that G_i non-amenable and H_i non-amenable, weakly amenable and with property (S).
If $p(L^\infty(X) \rtimes G)p \cong q(L^\infty(Y) \rtimes H)q$, then the actions are conjugate.

Recall: If $G = G_1 \times G_2$, then we say $G \curvearrowright (X, \mu)$ is irreducible if $G_i \curvearrowright (X, \mu)$ is ergodic for $i = 1, 2$.

Results - Strong solidity

Definition

A von Neumann algebra M is **strongly solid** if for all diffuse, amenable $A \subseteq M$ with expectation, the normaliser $\mathcal{N}_M(A)''$ remains amenable.

Theorem (Brothier-D-Vaes)

Let G be weakly amenable and with property (S). Suppose $L(G)$ is diffuse

(a) *$pL(G)p$ is strongly solid whenever $\mathrm{Tr}(p) < +\infty$*

(b) *If G has CMAP, then $L(G)$ is stably strongly solid (i.e. $L(G) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N}))$ is strongly solid)*

Examples

- (Houdayer-Raum, 2016 and Raum, 2016) criteria for groups G acting on trees such that $L(G)$ is non-amenable factor.

  In particular, Schlichting completion $BS(m, n)$ $(2 \leqslant |m| \leqslant n)$

  $L(G)$ strongly solid, non-amenable, type $III_{|m/n|}$

Contents

Dichotomy of (Popa-Vaes, 2014)

Theorem (Popa-Vaes, 2014)

Let Γ be countable, weakly amenable and in Ozawa's class $\mathcal{S}$ (e.g. Γ hyperbolic). Suppose $\Gamma \curvearrowright (B, \tau)$ trace-preserving. Let $M = B \rtimes \Gamma$. If $A \subseteq M$ is amenable relative to B, then

- *$\mathcal{N}_M(A)''$ remains amenable relative to B,*
- *OR*
- *$A \preceq_M B$.*

Proof of uniqueness of Cartan

- Suppose Γ non-amenable, weakly amenable and in Ozawa's class $\mathcal{S}$ and $\Gamma \curvearrowright (X, \mu)$ p.m.p., free and ergodic.
- Let $B = L^\infty(X)$ and $M = B \rtimes \Gamma$. Then, $\Gamma \curvearrowright B$ trace preserving.
- For $A \subset M$ arbitrary Cartan
  - ~~$\mathcal{N}_M(A)'' = M$ remains amenable relative to B,~~
  - OR
  - $A \preceq_M B \implies A = uBu^*$ for some $u \in \mathcal{U}(M)$ □

A more general dichotomy

Definition

Let M be a von Neumann algebra and G a group. A **co-action** is an injective, normal $*$-morphism $\Phi : M \to M \otimes L(G)$ such that

$$(\Phi \otimes 1)\Phi = (1 \otimes \Delta)\Phi,$$

where $\Delta : L(G) \to L(G) \otimes L(G)$ is the co-mult. given by $\Delta(u_g) = u_g \otimes u_g$.

Theorem (Brothier-D-Vaes)

Let (M, Tr) v.N.a. with semi-finite trace and $\Phi : M \to M \otimes L(G)$ a co-action. Suppose that G is weakly amenable and with property (S). Suppose $\mathrm{Tr}(p) < +\infty$. If $A \subseteq pMp$ is Φ-amenable, then

- *$\mathcal{N}_{pMp}(A)''$ remains Φ-amenable,*
- *OR*
- *A can be Φ-embedded.*

A more general dichotomy

Theorem (Brothier-D-Vaes)

Let (M, Tr) v.N.a. with semi-finite trace and $\Phi : M \to M \otimes L(G)$ a co-action. Suppose that G is weakly amenable and with property (S). Suppose $\mathrm{Tr}(p) < +\infty$. If $A \subseteq pMp$ is Φ-amenable, then

- *$\mathcal{N}_{pMp}(A)''$ remains Φ-amenable,*
- *OR*
- *A can be Φ-embedded.*

Definition

Let $A \subseteq pMp$ von Neumann subalgebra. Consider M-bimodule $\mathcal{H} = L^2(M) \otimes L^2(G)$ given by $x \cdot \xi \cdot y = \Phi(x)\xi(y \otimes 1)$

(a) A is Φ-embedded if $p \cdot \mathcal{H} \cdot p$ admits a non-zero A-central vector.

(b) A is Φ-amenable if $\mathcal{H}$ is left A-amenable, i.e. ∃ pos. lin. functional Ω on $\Phi(p)(Mp \otimes B(L^2(G)))$ that is $\Phi(A)$-central and such that $\Omega(\Phi(x)) = \mathrm{Tr}(x)$ for $x \in pMp$

Proof of uniqueness of Cartan subalgebra for lc groups

Theorem (Brothier-D-Vaes)

Let G be non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.

Proof.

- STP: $L(\mathcal{R})$ has unique Cartan subalgebra for $\mathcal{R}$ cross-section eq rel.
- Consider co-action $\Phi : L(\mathcal{R}) \to L(\mathcal{R}) \otimes L(G)$ given by

$$\Phi(f) = f \otimes 1, \qquad (\Phi(u_\varphi)F)(x_1, x_2, h) = F(\varphi^{-1}(x_1), x_2, g^{-1}h)$$

  for $f \in L^\infty(X)$, $\varphi \in [\mathcal{R}]$ and $F \in L^2(\mathcal{R}) \otimes L^2(G) = L^2(\mathcal{R} \times G)$. Here, g is such that $\varphi^{-1}(x_1) = gx_1$
- For $A \subseteq M$ Cartan subalgebra
  - ~~$\mathcal{N}_M(A)''$ remains Φ-amenable,~~
  - OR
  - A is Φ-embedded $\implies A = uL^\infty(Y)u^*$

□

Proof of strong solidity

Theorem (Brothier-D-Vaes)

Let G be weakly amenable and with property (S). Suppose $M = L(G)$ is diffuse. $pL(G)p$ is strongly solid whenever $\mathrm{Tr}(p) < +\infty$

Proof.

- Take $A \subseteq pMp$ diffuse, amenable
- Co-action: $\Delta : L(G) \to L(G) \otimes L(G)$ with $\Delta(u_g) = u_g \otimes u_g$
- By the dichotomy theorem

$\mathcal{N}_{pMp}(A)''$ remains Δ-amenable, $\implies \mathcal{N}_{pMp}(A)''$ is amenable

OR

~~A is Δ-embedded~~

□

Thank you for your attention!

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