Rigidity for von Neumann algebras given by locally compact groups

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Introduction

Action $G \curvearrowright (X, \mu)$ $\Longrightarrow$ Crossed product $L^\infty(X) \rtimes G$

- **Crossed product von Neumann algebra** $L^\infty(X) \rtimes G \subseteq B(L^2(X) \otimes L^2(G))$ generated by
  - copy of G: $u_g = \sigma_g \otimes \lambda_g$, where $(\lambda_g f)(h) = f(g^{-1}h)$ and $(\sigma_g a)(x) = f(g^{-1}x)$ for $f \in L^2(G)$, $a \in L^2(X)$,
  - copy of $L^\infty(X)$: $a \otimes 1$ for $a \in L^\infty(X)$.

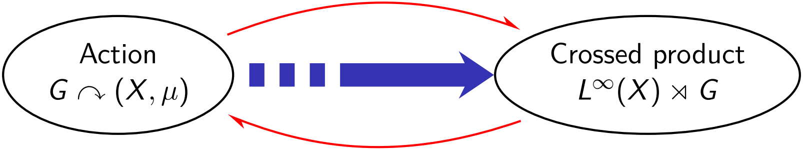
Note: $u_g a u_g^* = \sigma_g(a)$ for $a \in L^\infty(X)$

$\Rightarrow$ Algebra generated by $\{u_g\}_{g \in G}$ and $L^\infty(X)$:

$$A[G] = \text{span}\{a u_g\}_{a \in L^\infty(X), g \in G}$$

$$\Rightarrow \quad L^\infty(X) \rtimes G = \overline{A[G]}^{w.o.} = \overline{\text{span}\{a u_g\}_{a \in L^\infty(X), g \in G}}^{w.o.}$$

Introduction



Question

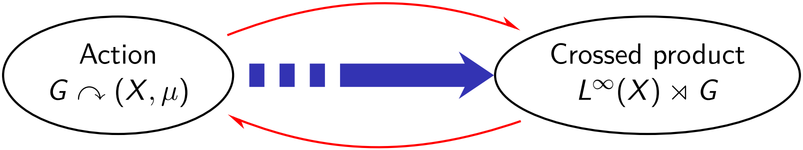
When is $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$?

- **(Connes, 1976)** All $L^\infty(X) \rtimes G$ are isomorphic for G discrete amenable

  e.g. S_∞, solvable groups

- **(Popa-Vaes, 2014)** $L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$

Introduction



Standing assumption

- **(essentially) free**: $\{x \in X \mid \exists g \in G : gx = x\}$ is a null set,
- **ergodic**: if $\mu(gA \Delta A) = 0$ for all $g \in G$, then A is null or co-null.
- **prob. measure preserving**: μ is prob. measure and $\mu(gA) = \mu(A)$.

⇒ $L^\infty(X) \times G$ is a factor

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Countable discrete group Γ

- $L^\infty(X) \rtimes \Gamma$ only depends on **orbit equivalence relation**

$$\mathcal{R}(\Gamma \curvearrowright X) = \{(gx, x) \mid x \in X, g \in \Gamma\}$$

⇒ Three "levels" of isomorphisms $\Gamma \curvearrowright (X, \mu)$, $\Lambda \curvearrowright (Y, \nu)$

Definition

$\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \nu)$ are

(a) **conjugate** if $\exists \varphi : \Gamma \xrightarrow{\sim} \Lambda$ and $\exists \theta : X \xrightarrow{\sim} Y$ such that $\theta(g \cdot x) = \varphi(g) \cdot \theta(x)$

(b) **orbit equivalent** if $\exists \theta : X \xrightarrow{\sim} Y$ such that $\theta(\Gamma x) = \Lambda \theta(x)$

(c) **W*-equivalent** if $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$

conjugate $\Gamma \curvearrowright (X, \mu) \cong \Lambda \curvearrowright (Y, \nu)$ $\Rightarrow$ orbit equivalent $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$ $\Rightarrow$ W*-equivalent $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$

Cartan subalgebras

Theorem (Singer, 1955)

If there exists an isomorphism

$$\Psi : L^\infty(X) \rtimes \Gamma \xrightarrow{\sim} L^\infty(Y) \rtimes \Lambda \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

then $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

- $L^\infty(X)$ is a **Cartan subalgebra**

Definition

$A \subseteq M$ is a **Cartan subalgebra** if

(i) A is maximal abelian (i.e. $A' \cap M = A$),

(ii) $\mathcal{N}_M(A) = \{u \in M \mid u \text{ unitary}, uAu^* = A\}$ generates M,

(iii) $\exists E : M \to A$ faithful, normal conditional expectation.

Cartan subalgebras

Theorem (Singer, 1955)

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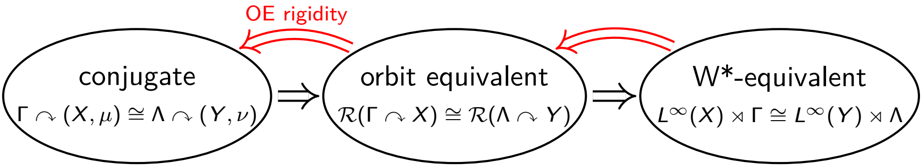
then $\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y)$.

- $L^\infty(X)$ is a **Cartan subalgebra**

  ⇒ if $L^\infty(X)$ is unique Cartan subalgebra (up to conjugacy)

$$\mathcal{R}(\Gamma \curvearrowright X) \cong \mathcal{R}(\Lambda \curvearrowright Y) \quad \Longleftrightarrow \quad L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$$



Uniqueness of Cartan subalgebras

$L^\infty(X) \rtimes \Gamma$ has unique Cartan (up to unitary conjugacy) if

- **(Ozawa-Popa, 2010)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Chifan-Sinclair, 2013)** Γ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ profinite
- **(Popa-Vaes, 2014)** $\Gamma = \mathbb{F}_n$ and $\Gamma \curvearrowright (X, \mu)$ arbitrary
- **(Popa-Vaes, 2014)** Γ hyperbolic and $\Gamma \curvearrowright (X, \mu)$ arbitrary

Theorem (Gaboriau, 2000)

$\mathcal{R}(\mathbb{F}_n \curvearrowright X) \ncong \mathcal{R}(\mathbb{F}_m \curvearrowright X)$ if $n \neq m$.

Corollary

$L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$.

Uniqueness of Cartan subalgebras

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Actually holds for Γ non-amenable, weakly amenable and in **Ozawa's class** $\mathcal{S}$

Definition

Γ belongs to Ozawa's class $\mathcal{S}$ if Γ is exact and $\exists \eta : \Gamma \to \text{Prob}(\Gamma)$ such that

$$\lim_{k \to \infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0$$

Contents

Non-discrete group G

- G locally compact, second countable and unimodular
- If G is locally compact, then $L^\infty(X)$ is not Cartan in $M = L^\infty(X) \rtimes G$

  BUT: $\exists$ **cross section** $Y \subseteq X$, i.e.

  (i) $\exists\, \mathcal{U} \subseteq G$ neighbourhood of $e \in G$ such that $\mathcal{U} \times Y \to X : (g, y) \mapsto gy$ is injective

  (ii) $G \cdot Y = X$ (up to null sets)

  Then, the **cross section equivalence relation**

$$\mathcal{R} := \mathcal{R}(G \curvearrowright X) \cap (Y \times Y) = \{(y, z) \in Y \times Y \mid \exists g \in G : gy = z\}$$

  is countable and

$$M \cong L(\mathcal{R}) \mathbin{\overline{\otimes}} B(\ell^2(\mathbb{N}))$$

➡ $L^\infty(Y) \mathbin{\overline{\otimes}} \ell^\infty(\mathbb{N})$ is Cartan in M

Contents

Results - uniqueness of Cartan

Definition

G has property (S) if $\exists$ a continuous map $\eta : G \to \text{Prob}(G)$ such that

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\|_1 = 0 \quad \text{uniformly on compact sets } g, h \in G.$$

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times \cdots \times G_n$ with G_i non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.

Results - uniqueness of Cartan

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Let $G = G_1 \times \cdots \times G_n$ with G_i non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.

Examples

Direct products of

- finite centre connected simple Lie groups of rank 1

  e.g. $\mathrm{SO}(n,1)$; $\mathrm{SU}(n,1)$; $\mathrm{Sp}(n,1)$
- automorphism groups of trees (and hyperbolic graphs).

Results - W*-rigidity

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$ be without compact normal subgroups. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free and irreducible. Suppose that G_i non-amenable and H_i non-amenable, weakly amenable and with property (S).
If $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$, then the actions are conjugate.

Recall: If $G = G_1 \times G_2$, then we say $G \curvearrowright (X, \mu)$ is irreducible if $G_i \curvearrowright (X, \mu)$ is ergodic for $i = 1, 2$.

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Dichotomy of (Popa-Vaes, 2014)

Theorem (Popa-Vaes, 2014)

Let Γ be countable, weakly amenable and in Ozawa's class $\mathcal{S}$ (e.g. Γ hyperbolic). Suppose $\Gamma \curvearrowright (B, \tau)$ trace-preserving. Let $M = B \rtimes \Gamma$. If $A \subseteq M$ is amenable relative to B, then

- *$\mathcal{N}_M(A)''$ remains amenable relative to B,*
- *OR*
- *$A \preceq_M B$.*

Proof of uniqueness of Cartan

- Suppose Γ non-amenable, weakly amenable and in Ozawa's class $\mathcal{S}$ and $\Gamma \curvearrowright (X, \mu)$ p.m.p., free and ergodic.
- Let $B = L^\infty(X)$ and $M = B \rtimes \Gamma$. Then, $\Gamma \curvearrowright B$ trace preserving.
- For $A \subset M$ arbitrary Cartan
  - ~~$\mathcal{N}_M(A)'' = M$ remains amenable relative to B,~~
  - OR
  - $A \preceq_M B \implies A = uBu^*$ for some $u \in \mathcal{U}(M)$

□

A more general dichotomy

Definition

Let M be a von Neumann algebra and G a group. A **co-action** is an injective, normal $*$-morphism $\Phi : M \to M \otimes L(G)$ such that

$$(\Phi \otimes 1)\Phi = (1 \otimes \Delta)\Phi,$$

where $\Delta : L(G) \to L(G) \otimes L(G)$ is the co-mult. given by $\Delta(u_g) = u_g \otimes u_g$.

Theorem (Brothier-D-Vaes)

Let (M, Tr) v.N.a. with semi-finite normal trace and $\Phi : M \to M \otimes L(G)$ a co-action. Suppose that G is weakly amenable and with property (S). Suppose $\mathrm{Tr}(p) < +\infty$. If $A \subseteq pMp$ is Φ-amenable, then

- *$\mathcal{N}_{pMp}(A)''$ remains Φ-amenable,*
- *OR*
- *A can be Φ-embedded.*

A more general dichotomy

Theorem (Brothier-D-Vaes)

Let (M, Tr) v.N.a. with semi-finite normal trace and $\Phi : M \to M \otimes L(G)$ a co-action. Suppose that G is weakly amenable and with property (S). Suppose $\mathrm{Tr}(p) < +\infty$. If $A \subseteq pMp$ is Φ-amenable, then

- *$\mathcal{N}_{pMp}(A)''$ remains Φ-amenable,*
- *OR*
- *A can be Φ-embedded.*

Definition

Let $A \subseteq pMp$ von Neumann subalgebra. Consider M-bimodule $\mathcal{H} = L^2(M) \otimes L^2(G)$ given by $x \cdot \xi \cdot y = \Phi(x)\xi(y \otimes 1)$

(a) A is Φ-embedded if $p \cdot \mathcal{H} \cdot p$ admits a non-zero A-central vector.

(b) A is Φ-amenable if $\mathcal{H}$ is left A-amenable, i.e. $\exists$ pos. lin. functional Ω on $\Phi(p)(Mp \otimes B(L^2(G)))$ that is $\Phi(A)$-central and such that $\Omega(\Phi(x)) = \mathrm{Tr}(x)$ for $x \in pMp$

Proof of uniqueness of Cartan subalgebra for lc groups

Theorem (Brothier-D-Vaes)

Let G be non-amenable, weakly amenable and property (S) and let $G \curvearrowright (X, \mu)$ be free, ergodic, pmp. Then, $L^\infty(X) \rtimes G$ has unique Cartan up to unitary conjugacy.

Proof.

- STP: $L(\mathcal{R})$ has unique Cartan subalgebra for $\mathcal{R}$ cross-section eq rel.
- Consider co-action $\Phi : L(\mathcal{R}) \to L(\mathcal{R}) \otimes L(G)$ given by

$$\Phi(f) = f \otimes 1, \qquad (\Phi(u_\varphi)F)(x_1, x_2, h) = F(\varphi^{-1}(x_1), x_2, gh)$$

  for $f \in L^\infty(X)$, $\varphi \in [\mathcal{R}]$ and $F \in L^2(\mathcal{R}) \otimes L^2(G) = L^2(\mathcal{R} \times G)$. Here, g is such that $\varphi^{-1}(x_1) = gx_1$
- For $A \subseteq L(\mathcal{R})$ Cartan subalgebra

  ~~$\mathcal{N}_M(A)''$ remains Φ-amenable,~~

  OR

  A is Φ-embedded $\implies A = uL^\infty(Y)u^*$

□

Thank you for your attention!

References I

- A. Connes, Classification of injective factors. Cases II_1 , II_∞, III_λ, $\lambda \neq 1$. **Ann. Math. 104** (1976), 73.
- I. Chifan and T. Sinclair, On the structural theory of II_1 factors of negatively curved groups. **Ann. Sci. Ecole Norm. S. 46** (2013), 1–33.
- D. Gaboriau, Coût des relations d equivalence et des groupes. **Invent. Math. 139** (2000), 41–98.
- N. Ozawa and S. Popa, On a class of II_1 factors with at most one Cartan subalgebra. **Ann. Math. 172** (2010), 713–749.
- S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of free groups. **Acta Math. 212** (2014), 141–198.

References II

S. Popa and S. Vaes, Unique Cartan decomposition for II_1 factors arising from arbitrary actions of hyperbolic groups. **J. Reine Angew. Math. 2014** (2014), 141–198.

I. M. Singer, Automorphisms of Finite Factors. **Am. J. Math. 77** (1955), 117–133.