What is... W*-rigidity?

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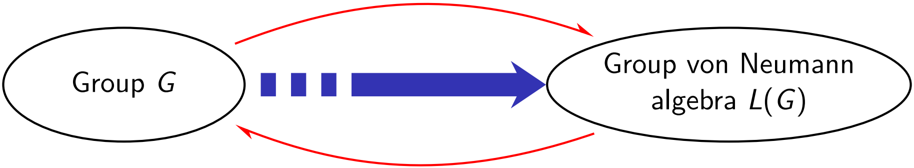
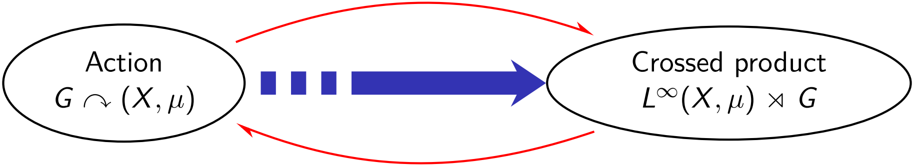
Introduction





Disclaimer

- This talk contains

Lies, white lies, downright lies, exaggeration and a tangled web of fraud and deception

Vaughan Jones

Contents

1 Von Neumann algebras
- Group von Neumann algebra
- Crossed product

2 W*-rigidity
- W*-rigidity for crossed products
- W*-rigidity for group von Neumann algebras

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Von Neumann algebras - Definition

- Introduced by **John von Neumann**
  - Motivated by quantum mechanics
- $\mathcal{H}$ Hilbert space
- $B(\mathcal{H})$ bounded operators
  - For $x \in B(\mathcal{H})$ and $\xi \in \mathcal{H}$

$$\|x\xi\| \leqslant M \|\xi\|$$

$$\|x\xi\| \leqslant \|x\| \|\xi\|$$

  - Every $x \in B(\mathcal{H})$ has an **adjoint** x^* satisfying

$$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle \qquad \text{for } \xi, \eta \in \mathcal{H}$$

Definition

A von Neumann algebra is a $*$-subalgebra $M \subseteq B(\mathcal{H})$ that is closed in the s.o. topology.

Von Neumann algebras - Examples

Definition

A von Neumann algebra is a $*$-subalgebra $M \subseteq B(\mathcal{H})$ that is closed in the s.o. topology.

Note: $x_i \to x$ in s.o. topology if and only if $\|x_i\xi - x\xi\| \to 0$ for $\xi \in \mathcal{H}$

Examples

- $B(\mathcal{H})$ (in part. $M_n(\mathbb{C})$)
- $L^\infty(X,\mu)$ (as subalgebra of $B(L^2(X,\mu))$
- The commutant A' of any set $A \subseteq B(\mathcal{H})$ closed under adjoint

  $\rightsquigarrow$ **von Neumann's bicommutant theorem** if $M \subseteq B(H)$ is a von Neumann algebra, then $M = (M')'$

Contents

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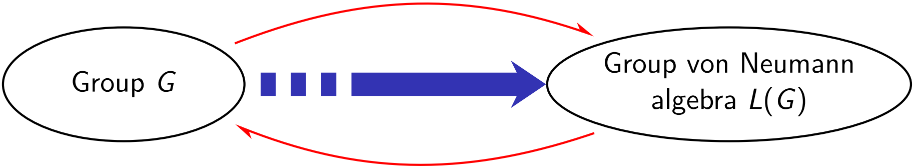
The group von Neumann algebra



- **Left-regular representation** $\lambda : G \to B(L^2(G))$

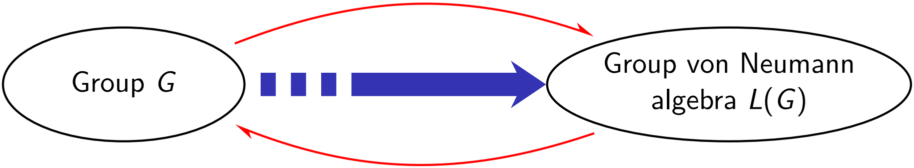
$$(\lambda_g f)(h) = f(g^{-1}h) \qquad \text{where } f \in L^2(G),\ g, h \in G$$

⇒ Group algebra $\mathbb{C}[G] = \text{span}\{\lambda_g\}_{g\in G}$

- **Group von Neumann algebra**:

$$L(G) = \overline{\mathbb{C}[G]}^{s.o.} = \overline{\text{span}\{\lambda_g\}_{g\in G}}^{s.o.}$$

The group von Neumann algebra



- **(Connes, 1976)** all $L(G)$ are isomorphic for G **amenable**
  - **e.g.** S_∞, solvable groups, ...
  - **non-e.g.** $\mathbb{F}_2$
- **(Murray and von Neumann, 1943)** $L(\mathbb{F}_2) \not\cong L(S_\infty)$
- **Open problem**: Is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ if $n \neq m$?

Two distinct group von Neumann algebras

- **(Murray and von Neumann, 1943)** $L(\mathbb{F}_2) \not\cong L(S_\infty)$

Idea of proof.

- In $L(S_\infty)$: property Γ
  - Define $g_n = (n, n+1) \in S_\infty$.
  - For $g \in S_\infty$, we have $g_n g = g g_n$ for n large

    ⇒ $\lambda_g \lambda_{g_n} = \lambda_{g_n} \lambda_g$ for n large

    ⇒ $x\lambda_{g_n} = \lambda_{g_n} x$ for $x \in \mathbb{C}[G]$ and n large

    ⇒ $(\lambda_{g_n})_n$ "asymptotically commutes" with every $x \in L(G)$, i.e.

$$\lambda_{g_n} x - x\lambda_{g_n} \to 0 \qquad \text{if } n \to \infty$$

- In $L(\mathbb{F}_2)$: no such sequence exists ✓ □

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Idea o

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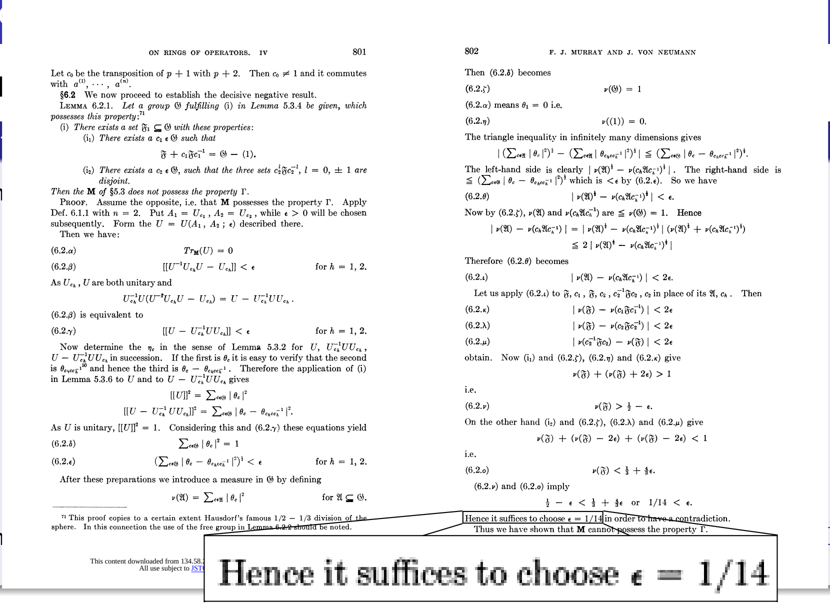


Let c_0 be the transposition of $p+1$ with $p+2$. Then $c_0 \neq 1$ and it commutes with $a^{(1)}, \cdots, a^{(n)}$.

§6.2 We now proceed to establish the decisive negative result.

LEMMA 6.2.1. *Let a group $\mathfrak{G}$ fulfilling* (i) *in Lemma* 5.3.4 *be given, which possesses this property:*[71]

(i) *There exists a set $\mathfrak{F}_1 \subsetneq \mathfrak{G}$ with these properties:*

(i₁) *There exists a $c_1 \in \mathfrak{G}$ such that*

$$\mathfrak{F} + c_1\mathfrak{F}c_1^{-1} = \mathfrak{G} - (1).$$

(i₂) *There exists a $c_2 \in \mathfrak{G}$, such that the three sets $c_2^l\mathfrak{F}c_2^{-l}$, $l = 0, \pm 1$ are disjoint.*

Then the **M** *of* §5.3 *does not possess the property* Γ.

PROOF. Assume the opposite, i.e. that **M** possesses the property Γ. Apply Def. 6.1.1 with $n = 2$. Put $A_1 = U_{c_1}$, $A_2 = U_{c_2}$, while $\epsilon > 0$ will be chosen subsequently. Form the $U = U(A_1, A_2; \epsilon)$ described there.

Then we have:

$$(6.2.\alpha) \qquad Tr_{\mathbf{M}}(U) = 0$$

$$(6.2.\beta) \qquad [[U^{-1}U_{c_h}U - U_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2.$$

As U_{c_h}, U are both unitary and

$$U_{c_h}^{-1}U(U^{-1}U_{c_h}U - U_{c_h}) = U - U_{c_h}^{-1}UU_{c_h}.$$

(6.2.β) is equivalent to

$$(6.2.\gamma) \qquad [[U - U_{c_h}^{-1}UU_{c_h}]] < \epsilon \qquad \text{for } h = 1, 2.$$

Now determine the η_c in the sense of Lemma 5.3.2 for U, $U_{c_h}^{-1}UU_{c_h}$, $U - U_{c_h}^{-1}UU_{c_h}$ in succession. If the first is θ_c it is easy to verify that the second is $\theta_{c_hcc_h^{-1}}$[50] and hence the third is $\theta_c - \theta_{c_hcc_h^{-1}}$. Therefore the application of (i) in Lemma 5.3.6 to U and to $U - U_{c_h}^{-1}UU_{c_h}$ gives

$$[[U]]^2 = \textstyle\sum_{c\in\mathfrak{G}} |\theta_c|^2$$

$$[[U - U_{c_h}^{-1}UU_{c_h}]]^2 = \textstyle\sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2.$$

As U is unitary, $[[U]]^2 = 1$. Considering this and (6.2.γ) these equations yield

$$(6.2.\delta) \qquad \textstyle\sum_{c\in\mathfrak{G}} |\theta_c|^2 = 1$$

$$(6.2.\epsilon) \qquad (\textstyle\sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2)^{\frac{1}{2}} < \epsilon \qquad \text{for } h = 1, 2.$$

After these preparations we introduce a measure in $\mathfrak{G}$ by defining

$$\nu(\mathfrak{A}) = \textstyle\sum_{c\in\mathfrak{A}} |\theta_c|^2 \qquad \text{for } \mathfrak{A} \subsetneq \mathfrak{G}.$$

[71] This proof copies to a certain extent Hausdorf's famous 1/2 − 1/3 division of the sphere. In this connection the use of the free group in Lemma 6.2.2 should be noted.

Then (6.2.δ) becomes

$$(6.2.\zeta) \qquad \nu(\mathfrak{G}) = 1$$

(6.2.α) means $\theta_1 = 0$ i.e.

$$(6.2.\eta) \qquad \nu((1)) = 0.$$

The triangle inequality in infinitely many dimensions gives

$$|(\textstyle\sum_{c\in\mathfrak{A}} |\theta_c|^2)^{\frac{1}{2}} - (\sum_{c\in\mathfrak{A}} |\theta_{c_hcc_h^{-1}}|^2)^{\frac{1}{2}}| \leqq (\sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2)^{\frac{1}{2}}.$$

The left-hand side is clearly $|\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}|$. The right-hand side is $\leqq (\sum_{c\in\mathfrak{G}} |\theta_c - \theta_{c_hcc_h^{-1}}|^2)^{\frac{1}{2}}$ which is $<\epsilon$ by (6.2.ε). So we have

$$(6.2.\theta) \qquad |\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}| < \epsilon.$$

Now by (6.2.ζ), $\nu(\mathfrak{A})$ and $\nu(c_h\mathfrak{A}c_h^{-1})$ are $\leqq \nu(\mathfrak{G}) = 1$. Hence

$$|\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})| = |\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}| (\nu(\mathfrak{A})^{\frac{1}{2}} + \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}})$$
$$\leqq 2 |\nu(\mathfrak{A})^{\frac{1}{2}} - \nu(c_h\mathfrak{A}c_h^{-1})^{\frac{1}{2}}|$$

Therefore (6.2.θ) becomes

$$(6.2.\iota) \qquad |\nu(\mathfrak{A}) - \nu(c_h\mathfrak{A}c_h^{-1})| < 2\epsilon.$$

Let us apply (6.2.ι) to $\mathfrak{F}, c_1$, $\mathfrak{F}, c_2$, $c_2^{-1}\mathfrak{F}c_2, c_2$ in place of its $\mathfrak{A}, c_h$. Then

$$(6.2.\kappa) \qquad |\nu(\mathfrak{F}) - \nu(c_1\mathfrak{F}c_1^{-1})| < 2\epsilon$$

$$(6.2.\lambda) \qquad |\nu(\mathfrak{F}) - \nu(c_2\mathfrak{F}c_2^{-1})| < 2\epsilon$$

$$(6.2.\mu) \qquad |\nu(c_2^{-1}\mathfrak{F}c_2) - \nu(\mathfrak{F})| < 2\epsilon$$

obtain. Now (i₁) and (6.2.ζ), (6.2.η) and (6.2.κ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) + 2\epsilon) > 1$$

i.e.

$$(6.2.\nu) \qquad \nu(\mathfrak{F}) > \tfrac{1}{2} - \epsilon.$$

On the other hand (i₂) and (6.2.ζ), (6.2.λ) and (6.2.μ) give

$$\nu(\mathfrak{F}) + (\nu(\mathfrak{F}) - 2\epsilon) + (\nu(\mathfrak{F}) - 2\epsilon) < 1$$

i.e.

$$(6.2.o) \qquad \nu(\mathfrak{F}) < \tfrac{1}{3} + \tfrac{4}{3}\epsilon.$$

(6.2.ν) and (6.2.o) imply

$$\tfrac{1}{2} - \epsilon < \tfrac{1}{3} + \tfrac{4}{3}\epsilon \quad \text{or} \quad 1/14 < \epsilon.$$

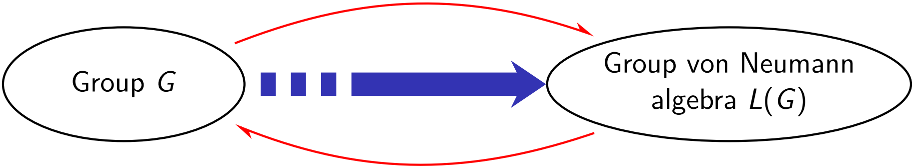
Hence it suffices to choose $\epsilon = 1/14$ in order to have a contradiction.

Thus we have shown that **M** cannot possess the property Γ.

Hence it suffices to choose $\epsilon = 1/14$

- In

The group von Neumann algebra



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  - **e.g.** S_∞, solvable groups, ...
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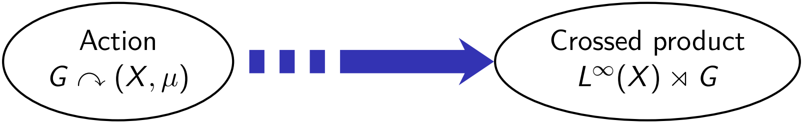
The crossed product



- $G \curvearrowright (X, \mu)$ induces action $G \curvearrowright^{\sigma} L^2(X)$ by

$$(\sigma_g a)(x) = a(g^{-1}x) \qquad \text{where } a \in L^2(X),\ g \in G,\ x \in X$$

- **Recall:** $G \curvearrowright^{\lambda} L^2(G)$

$$(\lambda_g f)(h) = f(g^{-1}h) \qquad \text{where } f \in L^2(G),\ g, h \in G$$

- Consider the following operators on $L^2(X \times G)$
  - $u_g = \sigma_g \times \lambda_g$

    ⇒ $\{u_g\}_{g \in G}$ copy of G
  - Copy of $L^\infty(X)$: $a \times 1$ for $a \in L^\infty(X)$

The crossed product $L^\infty(X,\mu) \rtimes G$

Action $G \curvearrowright (X,\mu)$ → Crossed product $L^\infty(X) \rtimes G$

- Consider the following operators on $L^2(X \times G)$
  - Copy of G: $u_g = \sigma_g \times \lambda_g$
  - Copy of $L^\infty(X)$: $a \times 1$ for $a \in L^\infty(X)$

Note: $u_g a u_g^{-1} = \sigma_g(a)$ for $a \in L^\infty(X)$ and $g \in G$

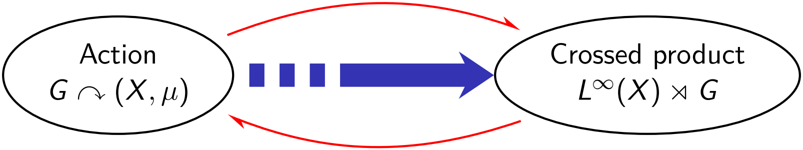
→ Algebra generated by $\{u_g\}_{g\in G}$ and $L^\infty(X)$:

$$A[G] = \text{span}\{a u_g\}_{a\in L^\infty(X), g\in G}$$

→ Crossed product von Neumann algebra

$$L^\infty(X) \rtimes G = \overline{A[G]}^{s.o.} = \overline{\text{span}\{a u_g\}_{a\in L^\infty(X), g\in G}}^{s.o.}$$

The crossed product $L^\infty(X,\mu) \rtimes G$



- $L(\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})) \cong L^\infty(\mathbb{T}^2) \rtimes L(\mathrm{SL}_2(\mathbb{Z}))$
  Note: $L(\mathbb{Z}) \cong L^\infty(\mathbb{T}^2)$
- **(Connes, 1976)** All $L^\infty(X) \rtimes G$ are isomorphic for G amenable
- **(Popa-Vaes, 2014)** $L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$

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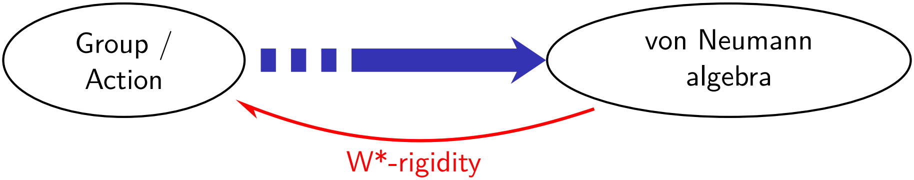
What is... W*-rigidity?



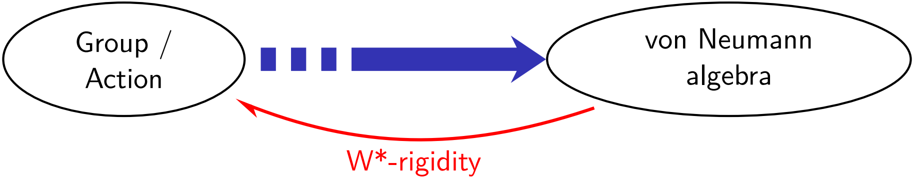
Examples of W*-rigidity

- **(Murray-von Neumann, 1943)** $L(\mathbb{F}_2) \ncong L(S_\infty)$
- **(Popa-Vaes, 2014)** $L^\infty(X) \rtimes \mathbb{F}_n \ncong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$

Non-examples of W*-rigidity

- **(Connes, 1976)** Amenable groups

What is... W*-rigidity?

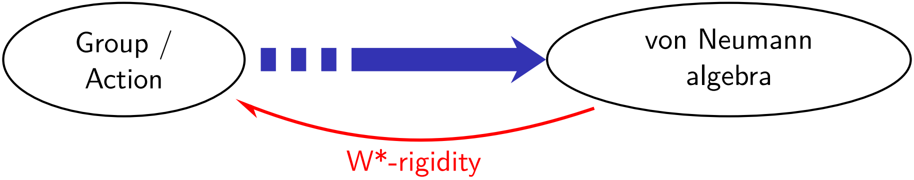


Definition

A **factor** is a von Neumann algebra with center $\mathcal{Z}(M) = \mathbb{C}1$.

- $L(G)$ is a factor for G countable, ICC
  - **ICC**: every conjugacy class (except $\{e\}$) is infinite
- $L^\infty(X) \rtimes G$ is a factor if $G \curvearrowright (X, \mu)$ is (essentially) free and ergodic
  - **(essentially) free**: $\{x \in X \mid \exists g \in G : gx = x\}$ is a null set
  - **ergodic**: if $\mu(gA\Delta A) = 0$ for all $g \in G$, then A is null or co-null

What is... W*-rigidity?



Standing assumptions

- G countable, usually ICC
- $G \curvearrowright (X, \mu)$ is free, ergodic and probability measure preserving.

Examples

- G compact group and $\Gamma \subseteq G$ countable, dense subgroup. Let $\Gamma \curvearrowright G$ by left-translation
- **Bernoulli action** G countable group, (X_0, μ_0) prob. space. Let $G \curvearrowright (X_0^G, \mu_0^{\otimes G})$ by

$$h \cdot (x_g)_{g \in G} = (x_{h^{-1}g})_{g \in G}$$

Contents

1 Von Neumann algebras
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W*-rigidity for crossed products

- $L^\infty(X) \rtimes G$ only depends on "orbit structure"

Definition

$G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are

(a) **conjugate** if $\exists \varphi : G \overset{\sim}{\to} H$ and $\exists \theta : X \overset{\sim}{\to} Y$ such that $\theta(g \cdot x) = \varphi(g) \cdot \theta(x)$

(b) **orbit equivalent** if $\exists \theta : X \overset{\sim}{\to} Y$ such that $\theta(Gx) = H\theta(x)$

(c) **W*-equivalent** if $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$

conjugate: $G \curvearrowright (X, \mu) \cong H \curvearrowright (Y, \nu)$ $\Rightarrow$ orbit equivalent: $G \curvearrowright (X, \mu) \overset{OE}{\sim} H \curvearrowright (Y, \nu)$ $\Rightarrow$ W*-equivalent: $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$

Cartan subalgebras

Theorem (Singer, 1955)

If there exists an isomorphism

$$\Psi : L^\infty(X) \rtimes G \xrightarrow{\sim} L^\infty(Y) \rtimes H \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

then $G \curvearrowright X$ is orbit equivalent to $H \curvearrowright Y$.

- $L^\infty(X)$ is a **Cartan subalgebra**

Definition

$A \subseteq M$ is a **Cartan subalgebra** if

(i) A is maximal abelian (i.e. $A' \cap M = A$),

(ii) $\mathcal{N}_M(A) = \{u \in M \mid u \text{ unitary}, uAu^* = A\}$ generates M,

Cartan subalgebras

Theorem (Singer, 1955)

If there exists an isomorphism

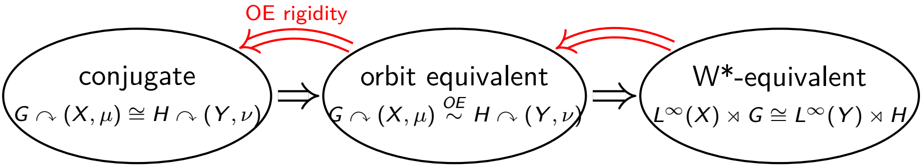
$$\Psi : L^\infty(X) \rtimes G \xrightarrow{\sim} L^\infty(Y) \rtimes H \quad \textit{satisfying } \Psi(L^\infty(X)) = L^\infty(Y),$$

then $G \curvearrowright X$ is orbit equivalent to $H \curvearrowright Y$.

- $L^\infty(X)$ is a **Cartan subalgebra**

  ⇒ if $L^\infty(X)$ is <u>unique</u> Cartan subalgebra

$$G \curvearrowright X \overset{OE}{\sim} H \curvearrowright Y \quad \Longleftrightarrow \quad L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$$



Uniqueness of Cartan subalgebras

$L^\infty(X) \rtimes G$ has unique Cartan if

- **(Ozawa-Popa, 2010)** $G = \mathbb{F}_n$ and $G \curvearrowright (X, \mu)$ profinite
- **(Chifan-Sinclair, 2013)** G hyperbolic and $G \curvearrowright (X, \mu)$ profinite
- **(Popa-Vaes, 2014)** $G = \mathbb{F}_n$ and $G \curvearrowright (X, \mu)$ arbitrary
- **(Popa-Vaes, 2014)** G hyperbolic and $G \curvearrowright (X, \mu)$ arbitrary

Theorem (Gaboriau, 2000)

$\mathbb{F}_n \curvearrowright X$ is not OE to $\mathbb{F}_m \curvearrowright Y$ whenever $n \neq m$.

Corollary

$L^\infty(X) \rtimes \mathbb{F}_n \not\cong L^\infty(Y) \rtimes \mathbb{F}_m$ if $n \neq m$.

W*-superrigidity for crossed products

Theorem (Popa, 2006)

Let G be ICC group and $G \curvearrowright (X, \mu)$ a Bernoulli action. Let H be a group with Property (T) and $H \curvearrowright (Y, \nu)$ arbitrary.
If $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$, then $G \curvearrowright (X, \mu)$ is conjugate to $H \curvearrowright Y$.

Definition

An action $G \curvearrowright (X, \mu)$ is **W*-superrigid** if any action $H \curvearrowright (Y, \nu)$ such that $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$ is conjugate to $G \curvearrowright (X, \mu)$.

- **(Peterson, 2010)** example of "virtually" W*-superrigid action
- **(Popa-Vaes, 2010)** family of W*-superrigid actions

W*-rigidity for crossed product of locally compact groups

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times \cdots \times G_n$ with

- *G_i connected, simple Lie group of rank 1,*
- *OR*
- *G_i automorphism group on a tree (or hyperbolic graph)*

Let $G \curvearrowright (X, \mu)$ be a free, ergodic action. Then, $L^\infty(X) \rtimes G$ has unique Cartan.

Theorem (Brothier-D-Vaes)

Let $G = G_1 \times G_2$ and $H = H_1 \times H_2$. Let $G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ be free and irreducible. Suppose that G_i are non-amenable and H_i as above. If $L^\infty(X) \rtimes G \cong L^\infty(Y) \rtimes H$, then the actions are conjugate

Note: If $G = G_1 \times G_2$, then we say $G \curvearrowright (X, \mu)$ is irreducible if $G_i \curvearrowright (X, \mu)$ is ergodic for $i = 1, 2$.

Contents

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2 W*-rigidity
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W*-rigidity for $L(G)$

Much less understood than crossed products

Problem

Is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$?

Connes rigidity conjecture (1980)

If G and H are ICC, property (T) groups then $L(G) \cong L(H)$ implies $G \cong H$.

Definition

G is **W*-superrigid** if for every group H, we have that $L(G) \cong L(H)$ implies $G \cong H$.

- (**Ioana-Popa-Vaes, 2013**) Example of W*-superrigid groups

Prime factors

Definition

A factor M is **prime** if $M \ncong M_1 \otimes M_2$ for (non-trivial) factors M_i.

Examples

- **(Ge, 1997)** $L(\mathbb{F}_n)$ is prime
- **(Ozawa, 2004)** $L(G)$ is prime for G hyperbolic
- **(Brothier-D-Vaes, 2018)** $L(G)$ is prime for certain (non-countable) automorphism groups of trees (or hyperbolic graphs)

Easy fact: $L(H_1 \times H_2) \cong L(H_1) \otimes L(H_2)$

Corollary

Let G be such that $L(G)$ is prime. If $L(G) \cong L(H)$, then $H \ncong H_1 \times H_2$.

Unique prime factorisation

Definition

Let $M_1, \ldots, M_n$ be prime factors. We say that $M_1 \otimes \cdots \otimes M_n$ has **unique prime factorisation** if for all prime factors $N_1, \ldots, N_m$ satisfying

$$M_1 \otimes \cdots \otimes M_n \cong N_1 \otimes \cdots \otimes N_m,$$

we have $m = n$ and $M_i \cong_s N_i$ for $i = 1, \ldots, n$ (after renumbering).

Examples

- **(Ozawa-Popa, 2004)** If $G_1, \ldots, G_n$ hyperbolic groups, then $L(G_1) \otimes \cdots \otimes L(G_n)$ has unique prime factorisation
- **(D, 2018)** Unique prime factorisation for certain non-countable groups

Unique prime factorisation

Corollary

Let $G = G_1 \times \cdots \times G_n$ be such that $L(G)$ has unique prime factorisation. If $H = H_1 \times \cdots \times H_m$ with $L(H_i)$ prime and such that

$$L(G) = L(G_1) \otimes \cdots \otimes L(G_n) \cong L(H_1) \otimes \cdots \otimes L(H_m) = L(H),$$

then $m = n$ and $L(G_i) \cong_s L(H_i)$ for $i = 1, \ldots, n$ (after renumbering).

Example

$L(\mathbb{F}_2 \times \mathbb{F}_2) \ncong L(\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2)$

Theorem (Chifan-de Santiago-Sinclair, 2016)

If $G = G_1 \times \cdots \times G_n$, where G_i ICC, hyperbolic. If $L(G) \cong L(H)$, then $H = H_1 \times \cdots \times H_n$.

Thank you for your attention!

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