Ozawa's class $\mathcal{S}$ for locally compact groups and unique prime factorization of group von Neumann algebras

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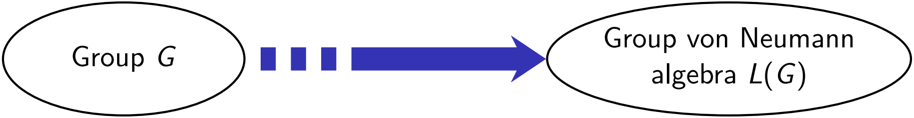
Skyline Communications

IPAM, Lake arrowhead, 2019

Group von Neumann algebra



- Consider left-regular representation $\lambda : G \to B(L^2(G))$

$$(\lambda_g \xi)(h) = \xi(g^{-1}h) \qquad g, h \in \Gamma,\ \xi \in L^2(G)$$

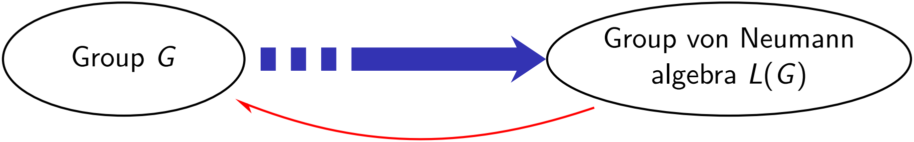
$\Rightarrow$ Group algebra $\mathbb{C}G = \text{span}\{\lambda_g\}_{g \in G}$

Definition

The **group von Neumann algebra** $L(G)$ is the von Neumann algebra generated by $\mathbb{C}G$, i.e.

$$L(G) = \overline{\mathbb{C}G}^{\text{w.o.}} = \overline{\text{span}\{\lambda_g\}_{g \in G}}^{\text{w.o.}}$$

Problem setting



Question

How much does $L(G)$ "remember" of the structure of G?

- **(Connes, 1976)** All $L(G)$ are isomorphic for G countable, amenable, icc
- **Open problem**: is $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ if $n \neq m$?

- **Ozawa's class $\mathcal{S}$**
  - G countable: **(Ozawa, 2004)**, **(Ozawa-Popa, 2004)**, ...
  - G locally compact: this talk

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Class $\mathcal{S}$ for countable groups

Γ countable group

Definition (Ozawa, 2006)

Γ is in **class** $\mathcal{S}$ (or is **bi-exact**) if Γ is exact and $\exists$ map $\eta : \Gamma \to \mathrm{Prob}(\Gamma)$ satisfying

$$\|\eta(gkh) - g \cdot \eta(k)\| \to 0 \qquad \text{if } k \to \infty$$

Examples

- Free groups $\mathbb{F}_n$

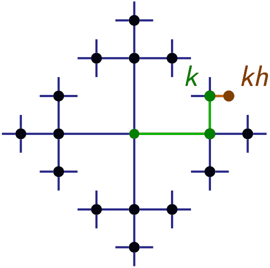
  $\eta(k) =$ unif. measure on path e to k

  - Right invariance ✓

    $\eta(kh) =$ unif. measure path e to kh

    $\eta(k) =$ unif. measure path e to k

  ➡ difference: path from k to kh



Class $\mathcal{S}$ for countable groups

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Examples

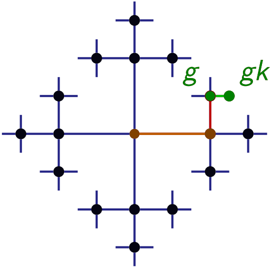
- Free groups $\mathbb{F}_n$

  $$\eta(k) = \text{unif. measure on path } e \text{ to } k$$

  - Left equivariance ✓

  $$\eta(gk) = \text{unif. measure path } e \text{ to } gk$$
  $$g \cdot \eta(k) = \text{unif. measure path } g \text{ to } gk$$

➡ difference: path from e to g



Class $\mathcal{S}$ for countable groups

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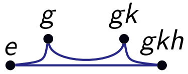
$$\|\eta(gkh) - g \cdot \eta(k)\| \to 0 \qquad \text{if } k \to \infty$$

Examples

- Free groups $\mathbb{F}_n$
- Amenable groups
  - $\exists$ sequence $\mu_n \in \text{Prob}(\Gamma)$
    $$\|\mu_n - g \cdot \mu_n\| \to 0$$
  - Define
    $$\eta(k) = \frac{1}{|k|} \sum_{i=|k|+1}^{|2k|} \mu_i$$



Class $\mathcal{S}$ for countable groups

Γ countable group

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Examples

- Free groups $\mathbb{F}_n$
- Amenable groups
- **(Adams, 1994)** Hyperbolic groups
- **(Skandalis, 1988)** Lattices in finite-center, connected, simple Lie groups with real rank 1

Exactness

Definition (Kirchberg-Wasserman, 1999)

Γ is **exact** if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

of Γ-C^*-algebras, also

$$0 \to A \rtimes_r \Gamma \to B \rtimes_r \Gamma \to C \rtimes_r \Gamma \to 0$$

is exact.

Examples

- Almost every group is exact

  e.g. amenable groups, hyperbolic groups, linear groups, countable subgroups of connected simple Lie groups, ...
- Examples of non-exact groups: (Gromov, 2003) and (Osajda, 2014)

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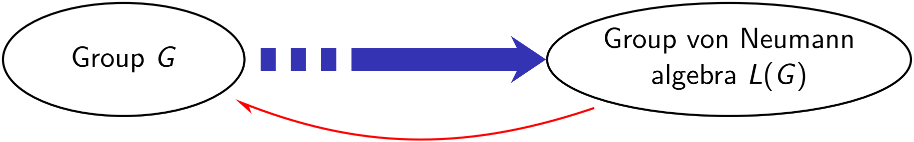
Applications



Theorem (Ozawa, 2004)

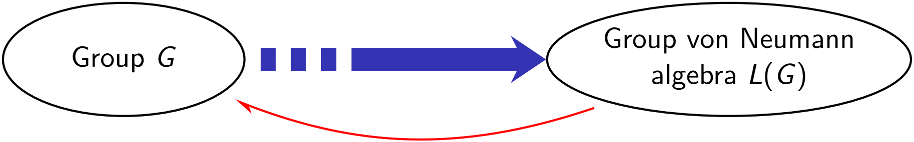
*$L(\Gamma)$ is **solid** if Γ is in class $\mathcal{S}$, i.e. for every diffuse $N \subseteq L(\Gamma)$ von Neumann subalgebra, the algebra $N' \cap L(\Gamma)$ is amenable.*

Corollary

*$L(\Gamma)$ is **prime** if Γ is non-amenable, icc and in class $\mathcal{S}$, i.e. $L(\Gamma) \ncong M_1 \mathbin{\overline{\otimes}} M_2$ if M_1, M_2 non-type I factors.*

$$\Rightarrow \quad L(\mathbb{F}_2 \times \mathbb{F}_2) = L(\mathbb{F}_2) \mathbin{\overline{\otimes}} L(\mathbb{F}_2) \ncong L(\mathbb{F}_2).$$

Applications



Theorem (Ozawa-Popa, 2004)

Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ with Γ_i non-amenable, icc and in class $\mathcal{S}$. Then $L(\Gamma) = L(\Gamma_1) \overline{\otimes} \ldots \overline{\otimes} L(\Gamma_n)$ has ***unique prime factorization (UPF)****, i.e. if*

$$L(\Gamma) = N_1 \overline{\otimes} \ldots \overline{\otimes} N_m$$

for prime factors $N_1, \ldots, N_m$, then $n = m$ and $N_i \cong_s L(\Gamma_i)$ (after relabeling).

$\Rightarrow$ $L(\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2) \not\cong L(\mathbb{F}_2 \times \mathbb{F}_2)$.

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Topological amenability – Definition

- G locally compact and second countable
- X compact topological space, $G \curvearrowright X$ continuous

Definition (Anantharaman-Delaroche, 1987)

$G \curvearrowright X$ is **(topologically) amenable** if $\exists$ weakly* continuous maps $\mu_n : X \to \mathrm{Prob}(G)$ such that

$$\|\mu_n(g \cdot x) - g \cdot \mu_n(x)\| \to 0$$

uniformly on X and on compact sets for $g \in G$.

Examples

- If $X = \{x_0\}$, then $G \curvearrowright X$ is amenable iff G is amenable
- If X discrete and $G \curvearrowright X$ free, then $G \curvearrowright X$ amenable

  $\mu_n(x) = \delta_x$

Topological amenability – Example

Definition (Anantharaman-Delaroche, 1987)

$G \curvearrowright X$ is **(topologically) amenable** if $\exists : \mu_n : X \to \mathrm{Prob}(G)$ of continuous maps such that

$$\|\mu_n(g \cdot x) - g \cdot \mu(x)\| \to 0$$

uniformly on X and on compact sets for $g \in G$.

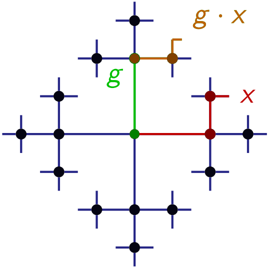
Examples

- $\mathbb{F}_2 \curvearrowright$ boundary of Cayley graph

$$\mu_n(x) = \begin{matrix}\text{unif. measure on first n} \\ \text{vertices of path } e \text{ to } x\end{matrix}$$

  - $\mu_n(g \cdot x) = (\ldots)$ path e to $g \cdot x$
  - $g \cdot \mu_n(x) = (\ldots)$ path g to $g \cdot x$

  ➡ difference: path from e to g



Topological amenability – Example

Definition (Anantharaman-Delaroche, 1987)

$G \curvearrowright X$ is **(topologically) amenable** if $\exists : \mu_n : X \to \text{Prob}(G)$ of continuous maps such that

$$\|\mu_n(g \cdot x) - g \cdot \mu(x)\| \to 0$$

uniformly on X and on compact sets for $g \in G$.

Examples

- $\mathbb{F}_2 \curvearrowright$ boundary of Cayley graph
- $\Gamma \curvearrowright$ boundary of Cayley graph for Γ hyperbolic

Characterization of class $\mathcal{S}$

Theorem (Ozawa, 2006)

A countable group Γ belongs to class $\mathcal{S}$ if and only if Γ has amenable action on a boundary that is $\mathit{small\ at\ infinity}$, i.e. $\exists$ compactification $h\Gamma$ of Γ such that

- *Actions by left and right translation extend to actions on $h\Gamma$,*
- *Action by right translation is trivial on $\nu\Gamma = h\Gamma \setminus \Gamma$,*
- *Action by left translation on $\nu\Gamma = h\Gamma \setminus \Gamma$ is topologically amenable.*

Link with C^*-algebras

Consider the following conditions:

(i) $G \curvearrowright X$ is amenable

(ii) $C(X) \rtimes G \cong C(X) \rtimes_r G$

(iii) $C(X) \rtimes_r G$ is nuclear

Theorem (Anantharaman-Delaroche, 1987)

For G countable, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$

Theorem (Anantharaman-Delaroche, 2002)

For G locally compact, we have $(i) \Rightarrow (ii) \Rightarrow (iii)$

Exactness and topological amenability

Definition

A group G is called **exact** if the operation of taking the reduced crossed product preserves short exact sequences.

Consider the following conditions

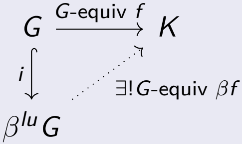
(i) G is exact,

(ii) $G \curvearrowright \beta^{lu} G$ is amenable,

(iii) $\mathrm{C}^*_r(G)$ is exact (i.e. taking minimal tensor product preserves exactness)

Definition

Left-equivariant Stone-Čech compactification $\beta^{lu} G$



$$C(\beta^{lu} G) \cong C_b^{lu}(G)$$
$$= \left\{ f \in C_b(G) \;\middle|\; \|\lambda_g f - f\|_\infty \to 0 \text{ if } g \to e \right\}$$

Exactness and topological amenability

Consider the following conditions

(i) G is exact,

(ii) $G \curvearrowright \beta^{lu} G$ is amenable,

(iii) $\mathrm{C}^*_r(G)$ is exact (i.e. taking minimal tensor product preserves exactness)

Theorem (Kirchberg-Wasserman, 1999; Ozawa, 2000)

For G countable, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$

Theorem (Anantharaman-Delaroche, 2002; Brodzki-Cave-Li, 2017)

For G locally compact, we have $(i) \Leftrightarrow (ii) \Rightarrow (iii)$.

- **Remark**: for locally compact $(iii) \Rightarrow (i)$ is open.

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Ozawa's class $\mathcal{S}$ for locally compact groups

Definition (Brothier-D-Vaes, 2018)

A locally compact group G is in **class** $\mathcal{S}$ if G is exact and $\exists$ continuous map $\eta : G \to \text{Prob}(G)$ satisfying

$$\lim_{k\to\infty} \|\eta(gkh) - g \cdot \eta(k)\| = 0$$

uniformly on compact sets for $g, h \in G$.

Examples

- Amenable groups
- **(Skandalis, 1988)** Finite-center, connected, simple Lie groups of real rank 1

  e.g. $\text{SL}_2(n, \mathbb{R})$, $\text{SO}(n,1)$, $\text{SU}(n,1)$, $\text{Sp}(n,1)$
- **(Brothier-D-Vaes, 2018)** Automorphism groups of trees and hyperbolic graphs

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Applications

Theorem (Brothier-D-Vaes, 2018)

*Let G be in class $\mathcal{S}$, then $L(G)$ is **solid**, i.e. for every diffuse $N \subseteq L(G)$ with expectation, we have $N' \cap L(G)$ is amenable.*

Corollary

*$L(G)$ is **prime** if G is in class $\mathcal{S}$ and $L(G)$ non-amenable factor*

Theorem (D, 2019)

Let $G = G_1 \times \cdots \times G_n$ with G_i locally compact groups in class $\mathcal{S}$ and $L(G_i)$ nonamenable factor. Then, $L(G) \cong L(G_1) \otimes \cdots \otimes L(G_n)$ has unique prime factorization, i.e. if

$$L(G) \cong N_1 \otimes \cdots \otimes N_m$$

with N_i prime, then $n = m$ and $L(G_i) \cong_s N_i$ (after relabeling).

Examples

Example (Suzuki)

- $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{F}_2$ by flipping generators

  ➡ $K = \prod_{n\in\mathbb{N}} \mathbb{Z}_2$ acts on $H = *_{n\in\mathbb{N}} \mathbb{F}_2$
- $G = H \rtimes K$ is in class $\mathcal{S}$ and $L(G)$ is nonamenable factor **(Suzuki, 2016)**

Corollary

$L(G) \ncong L(G \times G) \ncong L(G \times G \times G)$

Proof of unique prime factorization

Theorem (D)

Let $G = G_1 \times \cdots \times G_n$ with G_i in class $\mathcal{S}$ such that $L(G_i)$ is a nonamenable factor. Then, $L(G) \cong L(G_1) \mathbin{\overline{\otimes}} \ldots \mathbin{\overline{\otimes}} L(G_n)$ has UPF.

- Follows from combining
  - UPF results from **(Houdayer and Isono, 2017)** and **(Ando, Haagerup, Houdayer, and Marrakchi, 2018)**
  - Locally compact version of characterization of class $\mathcal{S}$

Theorem (D, 2019)

A locally compact group G belongs to class $\mathcal{S}$ if and only if it has amenable action on a compactification that is small at infinity, i.e. $\exists$ compactification $h^u G$ of G such that

- *Actions by left and right translation extend to actions on $h^u G$,*
- *Action by right translation is trivial on $h^u G \setminus G$,*
- *Action by left translation on $h^u G$ is topologically amenable.*

UPF results from (Houdayer-Isono, 2017)

Theorem (Houdayer-Isono, 2017; Ando, Haagerup, Houdayer, and Marrakchi, 2018)

A von Neumann algebra $M = M_1 \mathbin{\overline{\otimes}} \dots \mathbin{\overline{\otimes}} M_n$ has unique prime factorization if each M_i is a nonamenable factor satisfying strong condition (AO).

Definition (Houdayer-Isono, 2017)

A von Neumann algebra M with standard representation $(M, \mathcal{H}, J, P)$ satisfies **strong condition (AO)** if there exist C^*-algebras $A \subseteq M$ and $\mathcal{C} \subseteq B(\mathcal{H})$ such that

(i) A is exact and w.o. dense in M,

(ii) $\mathcal{C}$ is nuclear and contains A,

(iii) $[\mathcal{C}, JAJ] \subseteq \mathcal{K}(\mathcal{H})$

Proof of unique prime factorization

Theorem (D)

Let $G = G_1 \times \cdots \times G_n$ with G_i in class $\mathcal{S}$ such that $L(G_i)$ is a nonamenable factor. Then, $L(G) \cong L(G_1) \overline{\otimes} \ldots \overline{\otimes} L(G_n)$ has UPF.

Proof:

STP: Each $L(G_i)$ satisfies strong condition (AO)

- $A = \mathrm{C}^*_r(G_i)$ is exact and w.o. dense in $M = L(G_i)$
- $\mathcal{C} =?$
  - $G_i \curvearrowright h^u G_i$ is topologically amenable
    ⟹ $C(h^u G_i) \rtimes_r G_i \cong C(h^u G_i) \rtimes G_i$ is nuclear
  - Consider $\pi : C(h^u G_i) \rtimes G_i \to L^2(G_i)$ induced by covariant rep.
    $$g \mapsto \lambda_g, \qquad f \mapsto f|_{G_i} \qquad \text{for } f \in C(h^u G_i),\ g \in G_i$$
  - $\mathcal{C} = \pi(C(h^u G) \rtimes G_i)$
- $[\mathcal{C}, JAJ] \subseteq \mathcal{K}(\mathcal{H})$ ✓

□

New examples

Theorem (D, 2019)

Locally compact wreath products $B \wr_X^A H$ are in class $\mathcal{S}$ if B is amenable, H in class $\mathcal{S}$ and $H \curvearrowright X$ such that $\mathrm{Stab}_H(x)$ is amenable for all $x \in X$.

- **(Ozawa, 2006)** same result for discrete groups

Theorem (D, 2019)

Class $\mathcal{S}$ is closed under measure equivalence

- **(Sako, 2009)** same result for discrete groups

Thank you for your attention!

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